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Students' failure to understand fraction multiplication as part of a quantity

Yoppy Wahyu Purnomo[1*], Pasri[2], Tian Abdul Aziz[3], Masitah Shahrill[4], Irfan Wahyu Prananto[1]

[1]Department of Elementary School Education, Universitas Negeri Yogyakarta, Sleman, Indonesia
[2]Department of Mathematics Education, Universitas Muhammadiyah Prof. DR. HAMKA, Jakarta, Indonesia
[3]Department of Mathematics Education, Universitas Negeri Jakarta, Jakarta, Indonesia
[4]Sultan Hassanal Bolkiah Institute of Education, Universiti Brunei Darussalam, Gadong, Brunei Darussalam
*Correspondence: yoppy.wahyu@uny.ac.id

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Abstract

In the body of knowledge in mathematics education research, fractions are one of the researchers' concerns. The reason is because fractions are very difficult for students to understand. This study explores elementary school students' knowledge and obstacles in dealing with the multiplication of fractions. This study employs descriptive quantitative and qualitative approaches. Data were collected using the fractional knowledge test administered to 56 fifth-grade students and cognitive semi-structured interviews with six students depending on their test scores. The results of this study show that students' knowledge of fractions is restricted, with challenges interpreting context-based problems and the usage of "of" terms. Another finding shows that students' procedural knowledge is more dominant than conceptual knowledge. To develop students' knowledge of fractions, the portion of context-based learning must be an emphasis. The importance of developing research-based textbooks based on a suitable learning trajectory is highlighted.

Keywords: Conceptual Knowledge, Context-Based Problem, Elementary School, Multiplication of Fractions, Procedural Knowledge

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Researchers frequently express concerns regarding fractions as it is essential for algebraic thinking, proportional reasoning, advanced mathematics, and other science topics (Purnomo et al., 2021; Sidney et al., 2019). However, teaching and learning fractions are challenging for students (Li et al., 2022; Purnomo et al., 2019), prospective teachers (Baek et al., 2017; Rosli et al., 2020), and teachers (Copur-Gencturk, 2021; Putra, 2016). Without any underlying reasons, various mysterious rules have been presented in fractions teaching and learning. Skemp (1987) refers to these mysterious rules as rules without reason. In the growth of mathematics education research in elementary school, problems from cognitive, affective, and pedagogical perspectives and teaching resources linked to fractions have been significant (Purnomo et al., 2019).

Numerous studies on conceptual and procedural knowledge dealing with fractions have been undertaken (Hallett et al., 2010, 2012; Purnomo et al., 2017), fractions multiplication, specifically

(Khairunnisak et al., 2012; Simon et al., 2018; Son & Lee, 2016). However, little attention has been paid to students' emerging conceptualization of the meaning of fraction multiplication in studies that have focused on fraction multiplication knowledge (Simon et al., 2018). Therefore, this study examines students' conceptual and procedural knowledge when dealing with fraction multiplication problems with various meanings. We would also like to know where the obstacles and challenges are.

The variation in the meaning of the multiplication of fractions manifests in the multiplication of fractions problems that exist in everyday life. Therefore, students need to have literacy about these meaning variations to be flexible and effective in solving multiplication fraction problems. More than that, as problem solvers, students must also hold strong knowledge conceptually and procedurally in dealing with these variations of problems.

Focusing solely on fraction multiplication topic provides a more significant opportunity to yield a wide range of perspectives than focusing on a broader range of topics. Furthermore, we argue that research on how students learn about fractions is mostly dominated by Western perspectives. Even though research on students from various regions and demographics has not been widely explored, particularly in Southeast Asia. Therefore, this study provides diverse perspectives that complement previous studies and allow for the development of ideas for improving instructional activities, teaching materials, and curriculum.

The Meaning Fraction Multiplication

The context of fractions' multiplication is very familiar to students even before they go to formal school. They use it when interacting with the environment, even with non-mathematical language. For example, when a child asks for half of the four marbles that his brother has, it is mathematically written as 1/2 of 4. The same analogy applies to the multiplication of fractions as part of quantity and using fractions as operators. This example is familiar to students, and often they solve it mentally without going through rigid procedures. However, examples with the same concept involving more complex numbers can be found in the Lave (1988) study. The study by Jean Lave (1988) shows that adults can solve fraction multiplication problems in the kitchen (or in the supermarket) but not with the knowledge learned in schools. For instance, she asked dieters to have two-thirds of a portion of a litter of cottage cheese, which they found very practically in the kitchen by forking the cottage cheese out on the table, but they could not connect it to 2/3 × 3/4 (and strike out the threes).

The context of introducing fractional multiplication is familiar by relating it to the concept of multiplication as repeated addition. Children are also used to gaining this understanding and being introduced to examples in everyday life and formal mathematical contexts. By way of example, 2 packets of sugar with 1/2 kg each means 2 × 1/2 or adding 1/2 twice.

Multiplication of fractions has a different meaning than the multiplication of whole numbers, which is often defined as repeated addition. Simon et al. (2018) perceived that eight out of ten of Greer's (1992) multiplication and division situations were connected to the multiplication of fractions, so he opted to focus on five of them: rate, measure conversion, multiplicative comparison, part/whole, and multiplicative change. His study focused on how to improve the hypothetical learning trajectory for teaching fractions. In another study, Webel and DeLeeuw (2016) noted differences in the meaning of fraction multiplication and identified three of them: repeated addition, part of a part, and rectangular area. He also discusses the drawbacks of using the meaning of scaling rather than repeated addition.

We use the three definitions of fraction multiplication: multiplication as repeated addition, multiplication as a part of a quantity, and multiplication as a rectangular area. We split the definition of multiplication as a part of a quantity into three categories: part of a part, part of a whole, and a mix of both. As a result, we employ five different types of multiplication in our work. Table 1 shows an example of each of these constructs.

Table 1. The meaning of fraction multiplication

Subconstruct		Descriptions	Mathematical statement
Multiplication of fractions as repeated addition		This multiplication is like the multiplication of whole numbers. The main characteristic is that a multiplier is a whole number.	$a \times \frac{b}{c}$, ∀c ≠ 0
Multiplication of fractions as part of a quantity	Part of a whole	This multiplication is synonymous with finding the part of a quantity. Therefore, it is often marked with the term "of" and the multiplier is a fraction.	$\frac{a}{b}$ of c $= \frac{a}{b} \times c$, ∀b ≠ 0
	Part of a part		$\frac{a}{b}$ of $\frac{c}{d} = \frac{a}{b} \times \frac{c}{d}$, ∀b, d ≠ 0
	Part of a part of a whole		$\frac{a}{b}$ of $\frac{c}{d}$ of e $= \frac{a}{b} \times \frac{c}{d} \times e$, ∀b, d ≠0
Multiplication of fractions as a rectangular area		This multiplication is helpful in daily situations, such as determining the area of a rectangle. This multiplication approach is also helpful for bridging the division of fractions with a divisor that is numerically greater than the dividend.	$\frac{a}{b} \times \frac{c}{d}$, ∀b, d ≠ 0

Conceptual vs Procedural Knowledge

The typology of mathematical knowledge varies depending on the research's contexts and objectives and each researcher's perspective. Some studies divide knowledge into four dimensions: factual, procedural, conceptual, and metacognitive knowledge (Anderson & Krathwohl, 2001). On the other hand, several other studies categorised mathematical knowledge into two dimensions: conceptual and procedural knowledge (Hallett et al., 2012; Rittle-Johnson, 2017; Schneider & Stern, 2010; Star & Stylianides, 2013). In this study, we focus on two forms of knowledge: conceptual and procedural. Apart from the purpose of our research being to identify the phenomenon of the two, the dominance of the two is more commonly found in teaching resources related to the context of fractions.

Conceptual knowledge is the ability to integrate pieces of information to generate a full knowledge that underpins both explicit and implicit mathematical structures, whereas procedural knowledge is the understanding of how procedures function (Rittle-Johnson, 2017). Some scholars consider that gain of procedural knowledge must be preceded by gain of conceptual knowledge (Byrnes & Wasik, 1991), but most researchers agree that these should be intertwined (Kilpatrick et al., 2001). However, Rittle-Johnson et al. (2001) contend that procedural knowledge is conceivable and even widespread, in traditional educational environments, resulting in many students lacking proper conceptual knowledge. This teaching strategy assumes that children may learn the rules by rote without understanding the concepts implied in the procedure. As a result, procedural knowledge will be learned first, and then conceptual knowledge will be built on top of it (Rittle-Johnson et al., 2001). We contend that regardless of which is implemented first, both strategies aid students in solving mathematics problems. In practice, the relevance of context in interpreting both should be stressed so that children can make intelligent decisions about how best to use them.

In general, assessing procedural knowledge is simpler than conceptual knowledge (Rittle-Johnson, 2017). Almost every problem-solving activity involves procedural ability. Furthermore, Rittle-Johnson and Schneider (2014) stated that measuring conceptual knowledge includes implicit and explicit measures. The ability to evaluate unfamiliar procedures and concept examples, evaluate the quality of a given example in comparison to others, transform or represent mathematical objects, compare quantities, find principle-based quick procedures, solve key features, and sort examples into categories are all implicit measures. Meanwhile, describing a judgment, explaining why and how procedures work, and generating a concept map are examples of explicit measurements.

We used context-based problems to measure conceptual knowledge, while procedural knowledge is assessed using simple fractional multiplication operations. Students must understand problems and relevant information in context-based problems, analyse, and evaluate what decisions are appropriate when determining problem-solving methods in that context (Purnomo et al., 2019; Wijaya et al., 2015, 2018). Context-based problems are also required to connect mathematics to everyday life, which has various implications for understanding concepts, such as determining examples and non-examples.

We also used cognitive interviews to investigate the use of conceptual and procedural knowledge. It was applied to overcome weaknesses and unexpected things that may occur when the two forms of questions are applied, interviews are a solution to find out the dominant knowledge that arises in dealing with each question.

Research Questions

In this study, we focus on the two bits of knowledge, conceptual and procedural knowledge, when dealing with fraction multiplication problems in various meanings. The three questions guide our investigation: First, what are the characteristics of elementary school students' knowledge of fraction multiplication? Second, what factors contribute to elementary school students' knowledge of fraction multiplication? Third, what factors make it difficult for fifth-grade elementary school students to deal with fraction multiplication problems?

METHODS

Participants

A total of 56 fifth-grade students from one Jakarta public elementary school participated voluntarily in this study. They are 21 boys and 35 girls with the age range between 10 and 11 years old. Most of them were Javanese ethnic, followed by Sundanese, Betawi, Nusa Tenggara Timur, and Padang. Although the findings of this study were not to be generalized, with the diverse demographic characteristics in Jakarta, it could provide a complete landscape of the profile of elementary school students in Indonesia in dealing with the multiplication of fractions.

Indonesia's most recent mathematics curriculum requires students to learn fractions in second grade, that is, recognizing simple fractions such as 1/2, 1/3, and 1/4. They continue to learn more about fractions in third grade. In fourth grade, students learn how to add and subtract fractions, and then how to multiply and divide fractions in fifth grade. Therefore, we used grade 5 as our sample because they have studied the multiplication of fractions and other fraction concepts.

We administered a test to the participants on the first day of the project. The test examines students' knowledge of fraction multiplication. Subsequently, we conducted in-depth cognitive interviews based on the test score and teachers' considerations. As a result, two participants from each category were chosen: Soffa and Diyah for high scores, Mekar and Jeky for intermediate scores, and Olip and Elifa for low scores. These names are all pseudonyms.

Data Collection

Test

The test was based on five indicators that refer to three definitions of multiplication of fractions: (1) multiplication of fractions as repeated addition (one indicator), (2) multiplication of fractions as part of a quantity (two indicators), and (3) multiplication of fractions as rectangular area (one indicator) (Webel & DeLeeuw, 2016), and also multiplication of fractions as part of a part of a whole number (one indicator) (Son & Lee, 2016). Table 2 shows the five fraction multiplication test indicators and the order and distribution of the items.

Table 2. Test specification to assess knowledge of fraction multiplication

Indicator	Items	
	Conceptual knowledge	**Procedural knowledge**
Multiplication of fractions as repeated addition	1. Mr. Warno has three baskets of apples. Each basket weighs 3/4 kg. How does Mr. Warno know the total weight of the apples? Explain.	6. Solve: $3 \times \frac{1}{2}$
Multiplication of fractions as part of a whole number	2. Ridho has four chocolate bars. He will give 1/2 of the chocolates to his sister. How many chocolates bars did Ridho give to his sister? Explain.	9. Solve: $\frac{1}{2}$ of 3
Multiplication of fractions as part of a part	3. Ayu has a ½-meter long ribbon. She used ½ of the ribbon to make crafts. Explain how Ayu calculates the length of the ribbon she uses.	10. Solve: $\frac{1}{2}$ of $\frac{1}{2}$
Multiplication of fractions as part of a whole number	4. Juna has 12 candies. He gave ½ of the candies to Yoga. Then Yoga ate half of it. How many	7. Solve: $\frac{1}{2} \times \frac{1}{2} \times 6$

	sweets did Yoga eat? Explain.	
Multiplication of fractions as a rectangular area	5. Mr. Darma owns a rectangular rice farm with a length of $1\frac{1}{3}$ dam and a width of $1\frac{1}{2}$ dam. Explain how Mr. Darma calculates the area of his rice farm.	8. Solve: $1\frac{1}{2} \times 2\frac{1}{2}$

Table 2 depicts the two types of questions: (1) context-based problems which used to measure conceptual knowledge, and (2) short-form fractional multiplication operations questions examined participants' procedural knowledge. As a result, we distilled these five indicators into ten questions. The test was validated by researchers before being administered to six students to establish the readability of the questions, the level of difficulty, and the length of time to work. Thereafter, the test was administered to two classes on the same day for two hours.

Cognitive Interview

Cognitive interviews were used to investigate the answers given when taking the test and other possibilities arising from the questions. As a result, the interview protocol incorporated the ten questions from the fractional knowledge test.

Data Analysis

In this study, the following criteria were utilized to assess the validity of the data: 1) technical triangulation, 2) discussion and cross-checking, and 3) utilization of reference materials. Technical triangulation is accomplished by comparing data from the same source using multiple approaches—triangulation in this study was accomplished using two procedures, namely tests and interviews.

The test was administered to the 56 students that were present, and the results were assessed using the rubric. Because the highest possible score for each item is 4, and the lowest possible score is 0, the greatest possible score for this test is 20, and the lowest possible score is 0. A score of 4 is given to answers that use precise and complete concepts, terminology, mathematical notation, and calculation algorithms. Score 3 for answers that use precise and almost complete concepts, terminology, mathematical notation, and calculation algorithms. Score 2 for answers that use precise and incomplete concepts, terminology, mathematical notation, and calculation algorithms. Score 1 for answers that use inaccurate and incomplete concepts, terminology, mathematical notation, and calculation algorithms. Score 0 points for answers that reflect no knowledge or no work at all.

Following the analysis of the test findings, six students were drawn to participate as informants in the interview. Interviews were undertaken to validate the data derived from the test results to establish the data's trustworthiness. Interview recordings were transcribed and coded according to interesting patterns and referring to the main themes. The outcomes of the analysis were discussed and cross-checked by the researchers. To obtain valid and trustworthy patterns, discrepancies were identified and discussed.

RESULTS AND DISCUSSION

Result

Students' Knowledge of the Five Meanings of Fraction Multiplication

This section describes the students' work in interpreting the meaning of fraction multiplication. Figure 1 depicts the percentage of correct answers for each indicator.

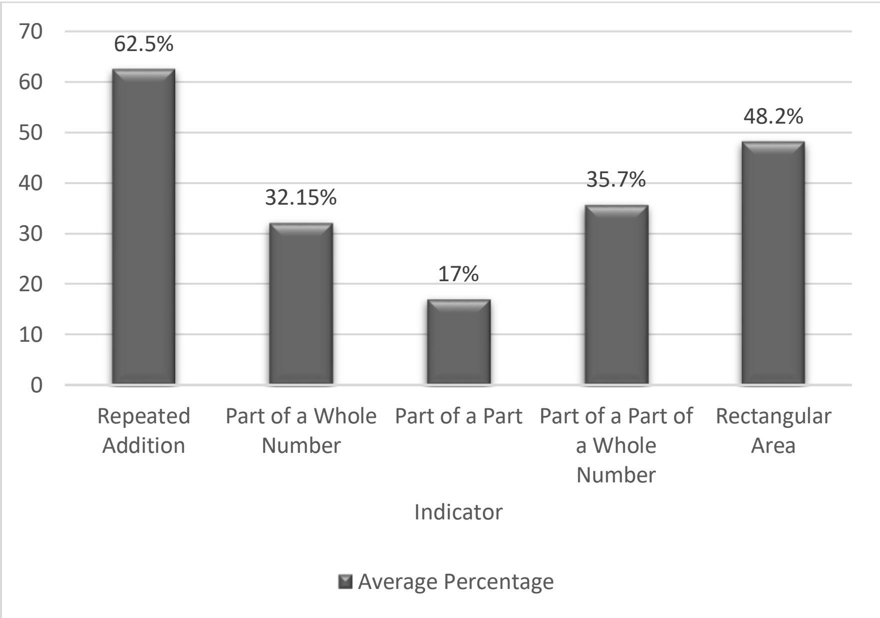

Figure 1. The percentage of correct answers based on the meaning of fraction multiplication indicators

Figure 1 indicates that the first indicator (i.e., questions 1 and 6) concerning the concept of multiplication of fractions as repeated addition may be completed by students with the conclusion that the answer is correct at most (i.e., 62.5 %). In this indication, 29 out of 56 students correctly answered the first question, while 41 out of 56 students correctly answered the sixth question. Here is an example of student work for this indicator.

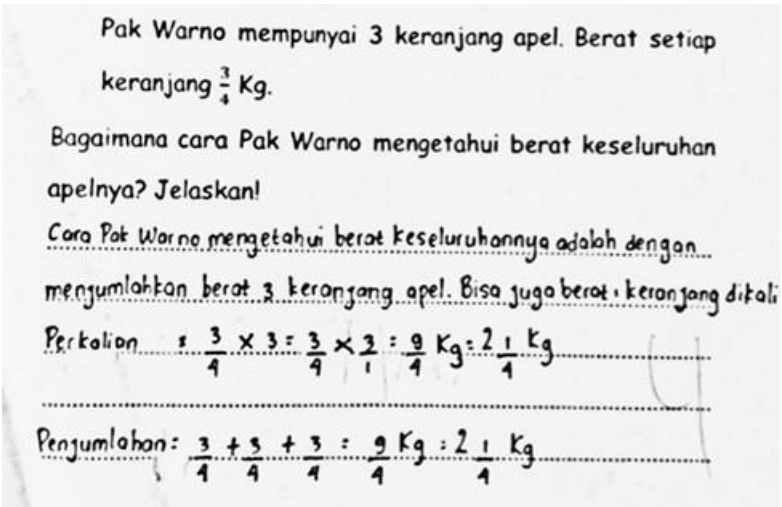

Figure 2. An example of student work for the case of multiplication as repeated addition

Translation:
Mr. Warno has three baskets of apples. Each basket weighs 3/4 kg. How does Mr. Warno know the total weight of the apples? Explain.
Answer:
Mr. Warno's way of knowing the total weight is by adding up the weight of 3 baskets of apples.
Multiplication: 3/4 × 3 = 3/4 × 3/1 = 9/4 Kg = $2\frac{1}{4}$ *Kg.*
Addition: 3/4 + 3/4 + 3/4 = 9/4 Kg = $2\frac{1}{4}$ *Kg.*

Figure 2 shows a sample of students' work on a problem about multiplication as repeated addition. She/he was more likely to have experience with similar problems since she/he was introduced to multiplication in their early grades. As a result, it stands to reason that this notion is more commonly accepted by their experience and receives the largest percentage of correct responses.

In contrast to the aforementioned facts, the indicator with the lowest correct answers (17%) was the third indicator, namely, the multiplication of fractions as part of a part (see Table 2). The third and the tenth questions belong to this indicator. The following figure holds examples of students' work for this indicator.

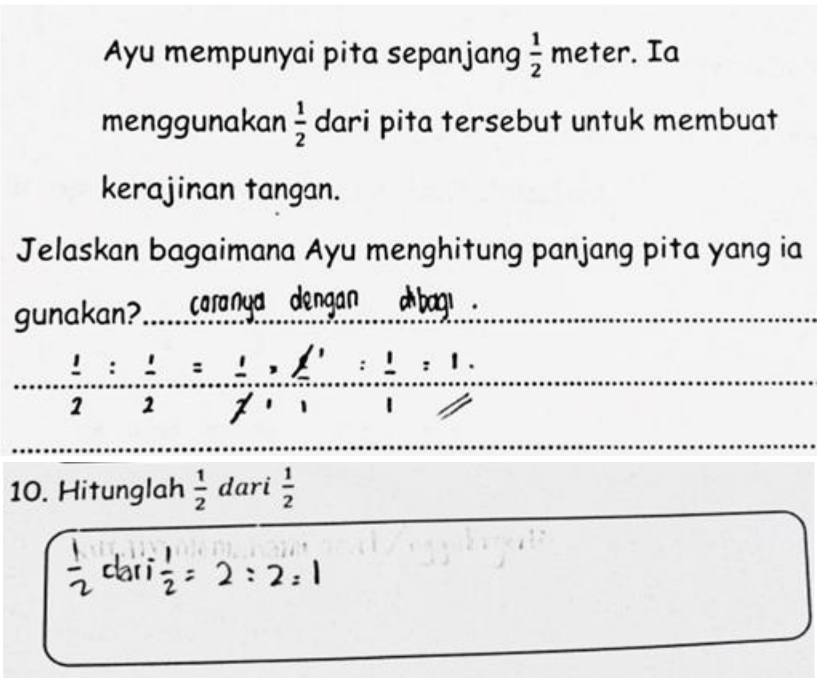

Figure 3. Two examples of students' work for the indicator of fractions as a part of a part

Translation:
Ayu has a ½-meter long ribbon. He used ½ of the ribbon to make crafts. Explain how Ayu calculates the length of the ribbon she uses.
Answer:
This problem is solved by dividing.
$\frac{1}{2}:\frac{1}{2}=\frac{1}{2}\times\frac{2}{1}=\frac{1}{1}=1.$

10. Solve: $\frac{1}{2}$ *of* $\frac{1}{2}$.

$\frac{1}{2}$ *of* $\frac{1}{2}$ *= 2 : 2 = 1.*

Figure 3 depicts students who do not comprehend the problem correctly and do not know what to do. This happened to many of the participants. Many were unfamiliar with the pattern of context-based problems and the use of the phrase "of" in fraction multiplication. The problem is difficult for students to understand, making it difficult to choose the proper mathematical operation to use. The problem is typically interpreted as a division situation, and it is resolved using the invert and multiply strategy.

Performance of Students on Conceptual and Procedural Knowledge

There were five conceptual knowledge questions and five procedural knowledge questions in the test. Conceptual knowledge questions started from number 1 to number 5, whereas procedural knowledge questions start from numbers 6 to number 10. The percentage of correct answers based on conceptual and procedural knowledge is depicted in Figure 4.

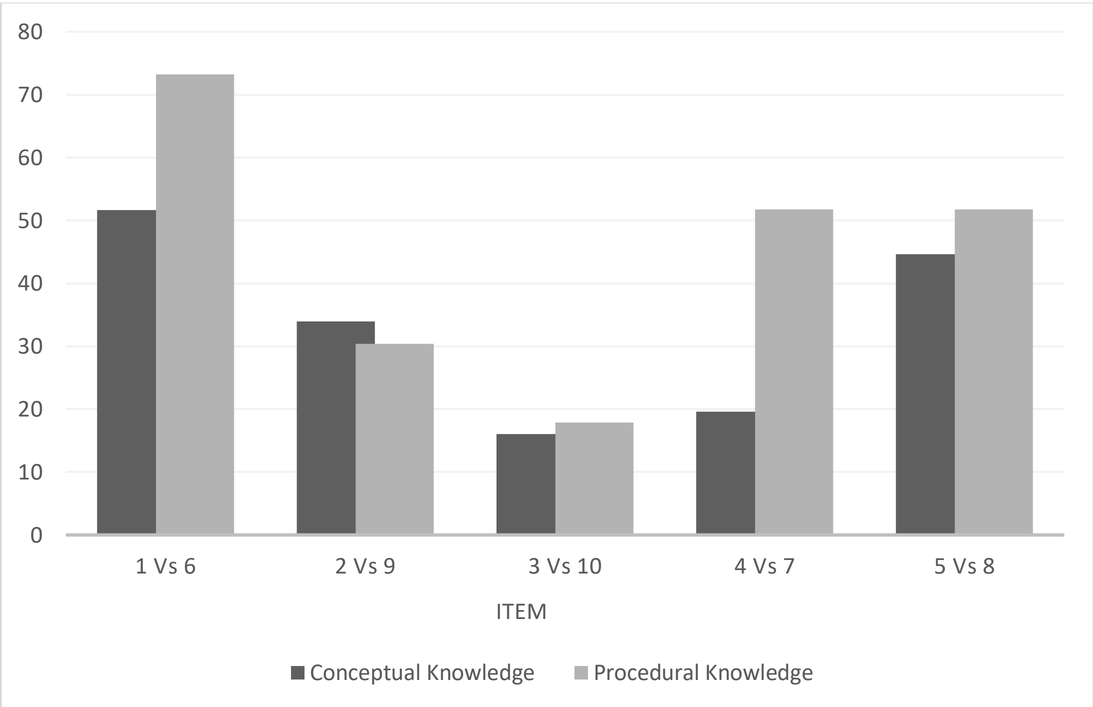

Figure 4. Percentage of correct answers to the conceptual and procedural knowledge questions

Figure 4 shows that, in general, students' procedural knowledge is stronger than their conceptual knowledge. According to the data, the average percentage of procedural knowledge (45%) is larger than conceptual knowledge (33%). Short-form questions are included in the procedural knowledge questions, and most of them, except for numbers 9 and 10, have been expressed in the form of multiplication operations symbols. As a result, students have no trouble comprehending and deciding the general solution processes, which consist of multiplying the numerator by the numerator and the denominator by the denominator. However, they encountered difficulties when the form of the multiplication operation was converted to the phrase "of," as in items 9 and 10. The following are three examples of their work that differ in how they interpret the two numbers of questions.

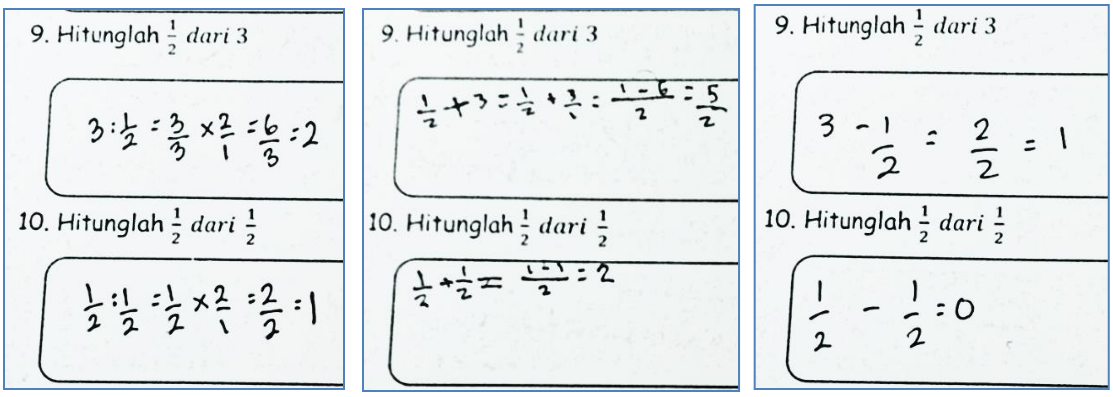

Figure 5. Various examples of students' answers to questions 9 and 10

Translation:

9. Solve: $\frac{1}{2}$ *of 3*

10. Solve: $\frac{1}{2}$ *of* $\frac{1}{2}$

Figure 5 shows that the students mistakenly define the term "of" one of the operations of fraction division, addition, and subtraction. It seems that they do not understand the meaning of the term "of" when determining a part of a part.

According to conceptual knowledge test, the meaning of the term "of" is also an obstacle for most students. As a result, they may encounter difficulty in dealing with context-based problems, particularly the problem of finding part of a part items 2, 3, and 4. Here are some examples of students' responses to question number 2.

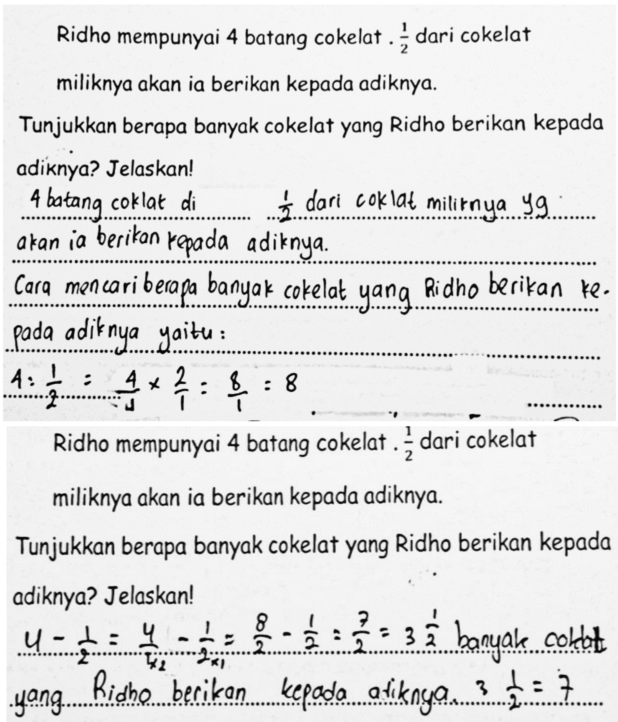

Figure 6. Examples of students' response to question no. 2

Translation:

Ridho has four chocolate bars. He will give 1/2 of the chocolate to his sister. How many chocolates did Ridho give to his sister? Explain.

- *This problem was solved by finding the number of chocolates that Ridho gave to his sister, namely 4 : 1/2 = 4/1 × 1/2 = 8/1 = 8.*
- *4 – 1/2 = 4/1 – 1/2 = 8/2 – 1/2 = 7/2 =* $3\frac{1}{2}$*. The number of chocolates that Ridho gave to his sister,* $3\frac{1}{2}$ *= 7.*

Figure 6 indicates that students make a mistake in determining correct mathematical operations to solve a context-based problem. They mistakenly interpret fraction multiplication as division and fraction subtraction. This finding is like the case 9 and 10 in which students encounter the term "of," representing the challenge of determining "part of a quantity."

Interesting cases occur in items 2 and 9. Both are cases of multiplying fractions as part of whole numbers. The result is that item 9, a representation of procedural knowledge, is higher than item 2, which represents conceptual knowledge. This is a unique case because most of the procedural knowledge dominates, but, in this case, the procedural knowledge is higher, although not too significant. We saw that item 9 uses the problem as "of" instead of the multiplication symbol. This became one of the impasses and was interpreted variously by students. As for number 2, that context may be very familiar to students in everyday life.

Case of High-Performing Student

Soffa and Diyah are among the high-performing students in the test. Soffa scored 38 out of 40, while Diyah scored 33. Based on their responses, they did not have significant difficulties completing the fraction multiplication test. Their score on conceptual knowledge items higher than on procedural understanding items. The detail results are presented in Table 3.

Table 3. Soffa and Diyah score on the test

Conceptual knowledge items			Procedural knowledge items		
Number	**Soffa**	**Diyah**	**Number**	**Soffa**	**Diyah**
1	4	4	6	4	4
2	4	2	7	4	4
3	4	3	8	2	4
4	4	4	9	4	4
5	4	4	10	4	0
Total	20	17	Total	18	16

Table 3 shows that Soffa did not face any significant challenges while working on the questions. However, in item 8, she made a calculation error. Meanwhile, Diyah encountered difficulties with multiplication as part of a quantity on both conceptual (i.e., items 2 and 3) and procedural aspects (i.e., item 10). Figure 7 represents Diyah's response to question number 2.

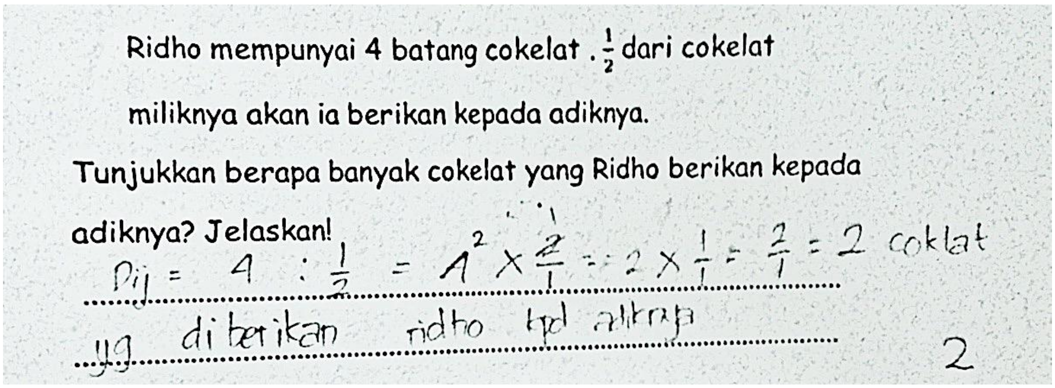

Figure 7. Diyah's response to question no. 2

Translation:
Ridho has four chocolate bars. He will give 1/2 of the chocolate to his sister. How many chocolates did Ridho give to his sister? Explain.
Answer:
The number of chocolates that Ridho gave to his sister:
$4 : \frac{1}{2} = 4 \times \frac{2}{1} = 2 \times \frac{1}{1} = \frac{2}{1} = 2$.

According to Figure 7, Diyah mistakenly used mathematical operations as the problem of fraction multiplication was solved using fraction division. When performing the fraction division operation using the scratch technique, she pairs the numerator and numerator to divide by the same number. This strategy should have happened before she used the invert and multiply strategy when performing the division operation.

Similar findings may be seen in Diyah's responses to an advance question during the interview. The following are excerpts from the interview and Diyah's work.

Researcher	: *I have a quarter of a loaf of bread. Diyah received half of mine. Diyah, how much bread do you get?*
Diyah	: *Diyah's bread slice is half of a quarter or one-eighth of a loaf.*
Researcher	: *How do you do that?*
Diyah	: *Like this (by pointing to Figure 8). Half of it is the same as being divided by 2, so if it's a quarter, it means that it is divided by 2 twice, making it 1/8.*

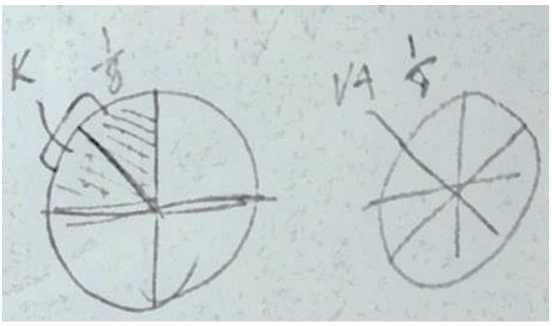

Figure 8. Diyah's work illustrates 1/2 : 1/4 in the interview session

Researcher : *Can you tell me how to write 1/2 of 1/4?*

Diyah : *1/2 of 1/4 equals 1/4 ÷ 1/2, so 1/4 divided by 2 equals 2 × 4/1, so, two by one equal 2.*

Based on the excerpt from the interview above, Dyah, with her conceptual knowledge, can illustrate that half of a quarter is one-eighth. When she was asked to write it down in mathematical sentences, Dyah related the context of "half of" as division by half but she was confused about whether to reverse the divisor. Hence, her answer contradicts the previous answer.

Case of the Intermediate-Performing Student

Based on the test results, Jeky and Mekar had a score of 27 and were in the intermediate category. Table 4 shows the distribution of their scores for each question.

Table 4. Jeki and Mekar's scores on the test

Conceptual knowledge items			Procedural knowledge items		
Number	**Jeki**	**Mekar**	**Number**	**Jeki**	**Mekar**
1	4	4	6	4	4
2	3	1	7	1	4
3	2	3	8	3	4
4	3	1	9	4	1
5	1	4	10	2	1
Total	13	13	Total	14	14

Table 4 shows that Jeki and Mekar had no trouble dealing with multiplication problems as repeated addition for both conceptual and procedural knowledge items. However, when it comes to multiplication as part of the quantity, the range of responses is inconsistent. Jeky's work on question 3 and Mekar's work on question 2 are shown in Figure 9.

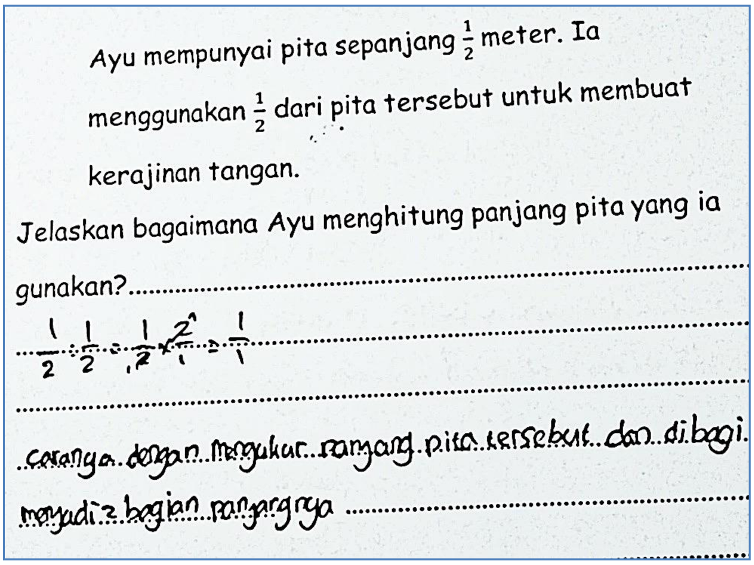

Figure 9. Jeky's response to question number 3

Translation:
Ayu has a ½-meter long ribbon. He used ½ of the ribbon to make crafts. Explain how Ayu calculates the length of the ribbon she uses.
$\frac{1}{2} : \frac{1}{2} = \frac{1}{2} \times \frac{2}{1} = \frac{1}{1}$
The solution is to measure the length of the tape and divide it into 2 parts.

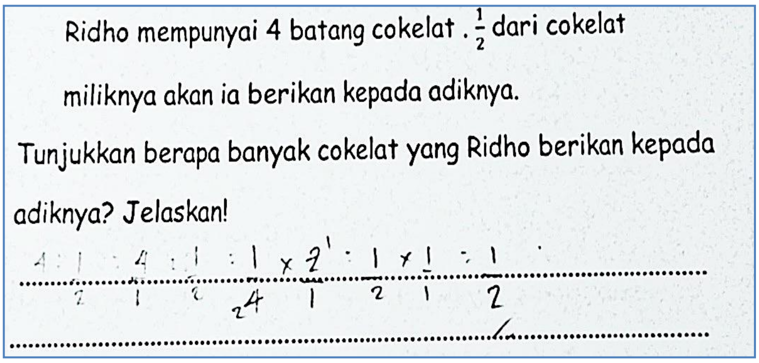
Ridho mempunyai 4 batang cokelat . $\frac{1}{2}$ dari cokelat miliknya akan ia berikan kepada adiknya.
Tunjukkan berapa banyak cokelat yang Ridho berikan kepada adiknya? Jelaskan!

Figure 10. Mekar's response to question number 2

Translation:
Ridho has four chocolate bars. He will give 1/2 of the chocolate to his sister. How many chocolates did Ridho give to his sister? Explain.
$4 : \frac{1}{2} = 4 : \frac{1}{2} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2} \times \frac{1}{1} = \frac{1}{2}.$

Based on Figure 9 and Figure 10, it is clear that Jeky and Mekar struggle with converting context-based problems into mathematical symbols. Both consider the meaning of "of" to be a case of division. However, the distinction is that Mekar sees the multiplication problem as a division problem and make a procedural error in solving the fraction division operation, despite having addressed this procedural error during the interview session.

During the interview, they show things that are not much different from what they did in the test. The following is an excerpt from the interview with Mekar regarding question number 2.

Researcher : *How much is half of the four chocolates that Brother has?*
Mekar : *Hmmmm... It is 4, because it is divided by half, which means 4 times 2/1, so it's eight.*

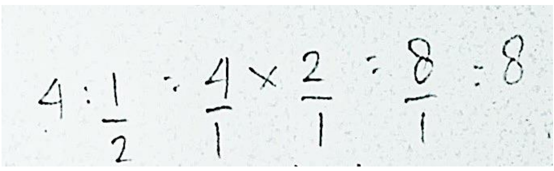

Figure 11. Mekar's work during the interview

Based on the findings of the interviews and the sample work in Figure 11, Mawar followed the procedure correctly but did not grasp the context. This pattern of responses appears in every question that contains the word "of" and is understood as a division case by both Mawar and Jeky. This work was also done by Dyah, a student with a high category, as previously discussed.

Case of the Low-Performing Student

Elifa and Olip are samples with low test scores, at 14 and 15, respectively. Table 5 shows the distribution of each score for each question.

Table 5. Elifa and Olip score on the test

Conceptual knowledge items			Procedural knowledge items		
Number	Elifa	Olip	Number	Elifa	Olip
1	1	1	6	1	4
2	1	1	7	1	1
3	2	1	8	2	1
4	1	4	9	2	0
5	1	2	10	2	0
Total	6	9	Total	8	6

According to Table 5, Elifa did not get perfect conceptual and procedural knowledge scores. However, Olip's performance is slightly better; he got score 4 when answering items number 4 and 6, but he got nothing when answering items 9 and 10. In other words, it seems that they have weaknesses in two areas: conceptual and procedural knowledge. Their lack of understanding is evident in their misinterpretation of questions and faults in operational processes. The following is an example of their effort to solve the multiplication problem as repeated addition.

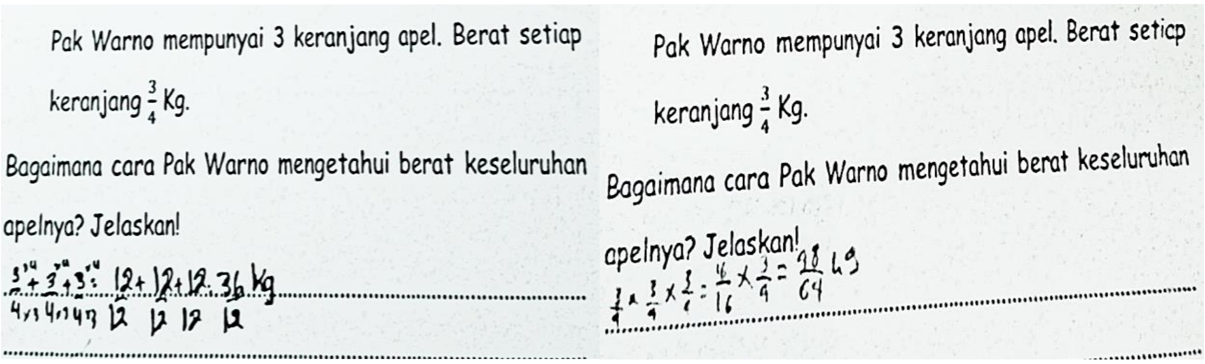

Figure 12. Elifa and Olip's responses to question number 2

Translation:

Mr. Warno has three baskets of apples. Each basket weighs 3/4 kg. How does Mr. Warno know the total weight of the apples? Explain.

Elifa's Answer: $\frac{3}{4}+\frac{3}{4}+\frac{3}{4}=\frac{12}{12}+\frac{12}{12}+\frac{12}{12}=\frac{36}{12}$

Olip's Answer: $\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}=\frac{16}{16}\times\frac{3}{4}=\frac{48}{64}$ *Kg.*

Elifa performs fraction addition by equating the denominators at first, as seen in Figure 12. She used his knowledge of fraction addition with different denominators to cope with fraction multiplication problems. Meanwhile, Olip made a mistake when converting the problem to mathematical symbols. These two types of mistakes, in addition to procedural problems, were commonly observed in the work of students in our sample.

We countered the test answers with interviews to reveal the two's incomplete understanding. The following is an excerpt from Elifa's interview.

Researcher : *1/2+1/3. How do you solve this?*
Elifa : *A third is multiplied by two in both the numerator and denominator, as well as for the half (see Figure 13).*

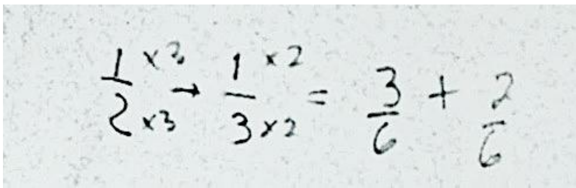

Figure 13. Elifa's work during the interview

According to the interview, Elifa had difficulty comprehending the basic principles of adding fractions. Elifa solves the issue in the same way for addition with the same denominator and addition with a different denominator, that is, by multiplying the numerator and denominator by the same value (cross multiplication strategy). Similarly, Elifa's comprehension of fraction multiplication can be noticed in her responses while working on short-form questions; Elifa's errors are apparent. For example, Elifa's responses to questions 6 and 10 are as follows.

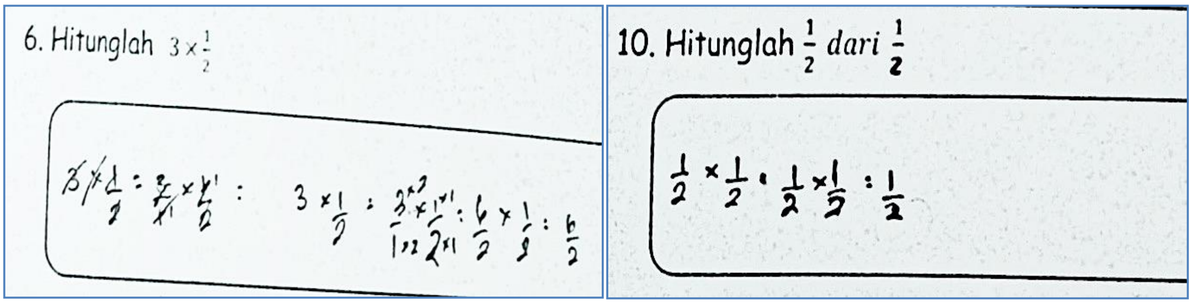

Figure 14. Elifa's response to questions number 6 and 10

Translation:

6. Solve: 3 × $\frac{1}{2}$

Answer: 3 × $\frac{1}{2}$ = $\frac{3}{1}$ × $\frac{1}{2}$ = $\frac{6}{2}$ × $\frac{1}{2}$ = $\frac{6}{2}$

10. Solve: $\frac{1}{2}$ of $\frac{1}{2}$

Answer: $\frac{1}{2}$ × $\frac{1}{2}$ = $\frac{1}{2}$ × $\frac{1}{2}$ = $\frac{1}{2}$

Figure 14 clearly shows that Elifa performed the multiplication operation incorrectly. Question 6 should have been answered with a multiplication procedure, but it was answered with an additional procedure, just as Elifa did with question 1 by cross multiplication. In question 10, Elifa multiplied the numerator, not the denominator. This example demonstrates that she lacks an understanding of the operation and procedures.

Olip falls into the low category based on test results in context-based problems and short-form questions. The following are her work for questions relating to question number 2.

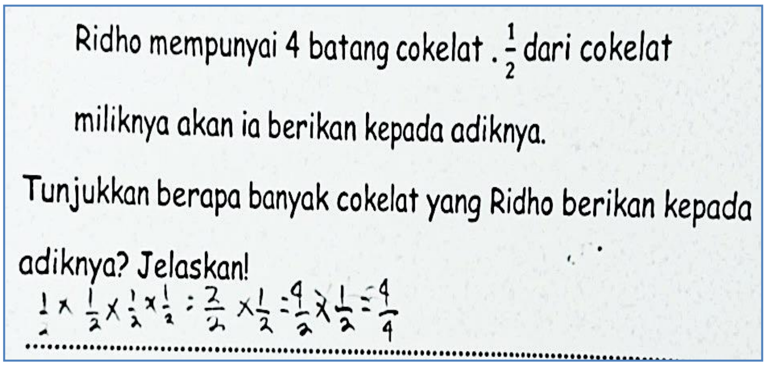

Figure 15. Olip's response to question number 2

Translation:
Ridho has four chocolate bars. He will give 1/2 of the chocolate to his sister. How many chocolates did Ridho give to his sister? Explain.
Answer: $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{2}{2} \times \frac{1}{2} = \frac{4}{2} \times \frac{1}{2} = \frac{4}{4}$.

According to Figure 15, Olip has trouble grasping the problems and techniques for solving fraction multiplication. Olip cross-multiplies the first and second fractions, then cross-multiplies again with the third fraction.

Based on the description above, it is possible to conclude that the factors that impede Elifa and Olip's understanding in dealing with the case of fraction multiplication include a lack of gain of the concept of fractions, misconceptions about context problems, incorrect calculation procedures, and a lack of mastery of the concept of addition as a prerequisite for learning multiplication.

Discussion

Procedural versus Conceptual Knowledge

According to the findings of this study, students in our sample are more likely to employ more procedural knowledge than conceptual knowledge. We noticed this dominance in the interview session in which they frequently employed various incorrect various algorithms and selected the incorrect operations. In other words, they are more used to dealing with mechanical problems than dealing with context-based problems. It is contrary to the objective of mathematics lesson, which is to develop student ability in solving context-based mathematical problems.

Rittle-Johnson (2017) contends that conceptual and procedural knowledge have an iterative bidirectional relationship, with advances in one enabling improvements in the other and procedural flexibility (Hallett et al., 2012). In Rittle-Johnson (2017), Canobi argues that conceptual knowledge contributes to the development, selection, and implementation of proper problem-solving procedures. Converse and applying procedures can help students develop and deepen their understanding of concepts, especially if it is designed to make the underlying concepts clearer. These two perspectives cannot be built independently.

The findings of our study show that procedural knowledge is more prevalent, but it is not built on or supported by conceptual knowledge. It contradicts with Rittle-Johnson and colleagues' previous statement. We are still unsure how to develop procedural knowledge and conceptual knowledge at the same time. We agree that students with strong conceptual knowledge have strong procedural abilities (Rittle-Johnson & Alibali, 1999), but not sure about the other way around.

According to previous research and experience an overemphasis on procedural knowledge may impede development of conceptual understanding and problem-solving abilities (Purnomo et al., 2014, 2019). This practice becomes a habit and makes sense when teachers focus on procedural knowledge and are satisfied because their focus is on results, rather than conceptual knowledge cultivation being something less effective due to time constraints and the completeness of the curriculum that is mandated for them to be completed (Purnomo et al., 2019).

Challenges to the Concept of Fractions Multiplication as Repeated Addition

According to the findings of this study, the definition of multiplication as repeated addition has been established for a more extended period than other fractional multiplication notions. This concept is related to the previously taught classical concept of whole-number multiplication. On the one hand, students will benefit from understanding the concept of multiplication of fractions which they are familiar with by connecting previous students' experiences.

On the other hand, fraction multiplication is not always regarded as repeated addition. In this type of multiplication, only whole numbers are used as multipliers. As a result, there is a knowledge gap between whole numbers and fractions, leading to the misconception that multiplication always yields the more larger result (Izsák, 2008; Khairunnisak et al., 2012; Webel & DeLeeuw, 2016). As a result, it is critical to broadening the concepts beyond the classic multiplication meaning. This obstacle is analogous to epistemic obstacles in which information that was previously learned and worked successfully in addressing problems at the time was out of sync with similar new concepts and preventing it from solving problems adequately (Brousseau, 2002; Tall, 2002). As the structure of mathematics is interrelated and involves a hierarchy, this issue is frequently faced in mathematics classes (Purnomo et al., 2014).

Bütüner (2021) contends that in his comparative study of Singaporean and Turkish mathematics textbooks, the presentation of various approaches to introducing fraction multiplication in Singapore textbooks provides students with more opportunities to learn with cover all meanings of multiplying fractions than simply presenting the classical meaning (repeated addition) of multiplication in Turkish textbooks. Furthermore, with the various approaches offered, students have more opportunities to reason and relate the concept of multiplication of fractions to other concepts of mathematics.

The Term "of" Refers to Fraction Multiplication

Our findings highlight the use of the term "of" ("dari" in Bahasa Indonesia) which is unfamiliar to students. Students have difficulties in locating the appropriate mathematical operation to represent the term "of" in the problem. They are frequently interpreted as division cases and solved using invert and multiply strategies.

We hypothesize that this obstacle arises from misinterpreting the word's meaning. In our sample, Mekar interprets 'half of four' as '4 ÷ ½'. Aside from word meanings, we can see that the context of multiplication, which is closely related to or the inverse of division, frequently confused the two students. This finding appears most frequently in students' responses to questions 2 and 3 which were about part of a quantity. This is in contrast to the findings of the Purnomo et al. (2019) study. They interviewed three elementary school students after administering a fraction division test. When asked to divide two by 1/2, one respondent counts the number of times 1/2-an equals two or uses repeated addition or multiplication. This is common in various situations that allow children to feel at ease with division as the inverse of multiplication, multiplication as the inverse of division, and the habit of using the invert and multiply strategy—however, many students fail in this process.

According to Webel's research on textbooks used in Korea and America, textbooks in Korea use the term part of a fractional component, whereas textbooks in America use two terms to express fraction multiplication, namely sign times (×) and the term "part of the fraction" (Webel & DeLeeuw, 2016). In contrast to the two Countries, the term "multiplication of fractions" emphasizes the symbol (×) in the process of introducing fractional multiplication in Indonesia. This means students are constrained when faced with questions that use the term "of". (Based on the response to questions number 9 and 10).

CONCLUSION

This study has at least a few significant aspects that require additional investigation. Each of these elements is inextricably linked to the others. First, we identified that students' knowledge of fractions was dominated by procedural knowledge but weak in conceptual knowledge. Students' knowledge that focuses on procedural knowledge and is not supported by conceptual knowledge would make it challenging to analyse the likely locations of errors committed and identify the appropriate strategy for addressing issues, particularly non-routine situations. Second, we identified that students' knowledge regarding multiplication as repeated addition was well structured and supported students to work with certain fractions multiplication. However, on the other hand, this understanding limits them from solving multiplication fractions in non-repeated addition contexts. Third, we identified that students have not had many opportunities to engage with fraction multiplication with various meanings, particularly the meaning of fraction multiplication as a part of a quantity.

We believe that conceptual knowledge should be prioritized for success in mathematics and its future applications. This perspective, however, is only based on our observation that high procedural knowledge does not apply to their conceptual knowledge, although high conceptual knowledge considerably adds to the application of procedural knowledge. Future studies should examine adequate evaluation to determine which knowledge is more desired and must first be generated for mathematical achievement and its future and everyday usage. It is also necessary to consider the teacher's time constraints and curricular constraints.

The researcher realizes that the limitations of this study need to be conveyed to be of concern both practically and theoretically. We identify two aspects that need attention. First, the messy construct of measures of conceptual and procedural knowledge. Our research measures conceptual knowledge using context-based problems, assuming that through the context, students need the ability to understand problems, select relevant information, and analyse and evaluate what decisions are appropriate in determining problem-solving methods in that context. Future studies need to accommodate explicit and implicit measures of conceptual knowledge as proposed by Rittle-Johnson and Schneider (2014). Additionally, since procedural knowledge is simpler to assess than conceptual knowledge, we use simpl e fractional multiplication operations (Rittle-Johnson, 2017). We also assessed it by cognitive interview, so the knowledge that emerged could be identified. Second, the term "of," which in this study we associate with the meaning of fraction multiplication, "finding a part (a fraction) of the quantity," has semantic hurdles that need to be considered. Although there is no problem in Bahasa Indonesia, in English, the term "of" often has multiple meanings and may not be fully translatable between languages. Therefore, it is necessary to consider the context of the accompanying problems.

Finally, the importance of developing research-based textbooks based on a suitable learning trajectory is highlighted. Therefore, future research needs to focus on this aspect because textbooks are the primary source for teachers in guiding their teaching and learning practices in the classroom. As a result of research-based textbooks, research-based practice will emerge.

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Declarations

Author Contribution	:	YWP: Conceptualization, Methodology, Writing - Review & Editing, Validation, Supervision, and Visualization; P: Formal analysis, Investigation, Writing - Review & Editing, TAZ: Validation, Writing - Original Draft; MS: Writing - Review & Editing; and IWP: Writing - Review & Editing.
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