Journal on Mathematics Education
Volume 15, No. 1, 2024, pp. 55-78

Transformation geometry in eleventh grade using digital manipulative batik activities

Sani Sahara[1,*], Maarten Dolk[2], Agus Hendriyanto[1], Tri Atmojo Kusmayadi[3], Laila Fitriana[4]

[1]Postgraduate of Mathematics Education, Universitas Pendidikan Indonesia, Bandung, Indonesia
[2]Inspiration4Learning, Utrecht, The Netherlands
[3]Department of Mathematics, Universitas Sebelas Maret, Surakarta, Indonesia
[4]Department of Mathematics Education, Universitas Sebelas Maret, Surakarta, Indonesia
*Correspondence: saharasani@gmail.com

Received: 6 March 2023 | Revised: 6 October 2023 | Accepted: 14 October 2023 | Published Online: 1 November 2023

Abstract

Learning Transformation Geometry (TG) needs a more informal approach to concern situational problems. This study aims to develop actionable knowledge of TG in the form of design related to context and yet general enough to use digital manipulative activities in new situations. We propose such knowledge in the form of conjectured Local Instructional Theory (LIT) in the framework of design research methodological framework. The designed learning activities were based on Realistic Mathematics Education (RME) principles and used batik as the context and van Hiele's mode of geometric thought. In addition, the CorelDraw software is used as a tool to transform batik-making activities into a digital manipulative environment. The design consists of a pre-assessment and four learning activities. The data were analyzed retrospectively in accordance with the HLT. The analysis of the data described above and the justification of the processes during the teaching experiment indicate a compelling trajectory for students learning transformation geometry for this specific context and the prospect for future studies.

Keywords: Batik, Design Research, Local Instructional Theory, Realistic Mathematics Education

How to Cite: Sahara, S., Dolk, M., Hendriyanto, A., Kusmayadi, T. A., & Fitriana, L. (2024). Transformation geometry in eleventh grade using digital manipulative batik activities. *Journal on Mathematics Education, 15*(1), 55-78. http://doi.org/10.22342/jme.v15i1.pp55-78

One of the objectives of mathematics education held in the Indonesian curriculum was to develop an attitude of appreciation of the value of mathematics in life, including curiosity, attention, and interest in learning mathematics (Depdiknas, 2006). The framework has changed as the Kurikulum Merdeka (KM or emancipated curriculum) takes over. The subject of mathematics is now more aimed at developing the independence and creativity of students (Pratikno et al., 2022; Zidan & Qamariah, 2023), which is relevant to the goal of the student's profile (Pelajar Pancasila) (Walukow et al., 2023). The aim justifies the rational matter that mathematics equips students with ways of thinking and reasoning that form their continuous thought processes, leading to the formation of a flow of understanding toward the learning material (Hasanah, 2023). Amongst mathematics subjects in the KM is geometry, in which knowledge about transformation geometry (TG) is discussed.

"A stretch along the royal road to geometry," as Martin (2012) inferred, TG connects geometry and algebra. In this manner, TG is described as an aspect of geometry that transforms geometrical objects into various images under reflection, translation, rotation, glide reflection, and magnification on a plane (Evbuomwan, 2013). Foremost, the transformation geometry is not just a mere change of shapes but widely, also a dynamic approach to learning (Hollebrands, 2003). Among countries, TG is organized in the mathematics school curricula (Mashingaidze, 2012). The United States's National Council Teaching Mathematics (NCTM, 2020) emphasizes how teaching TG should assist students in applying transformations and using symmetry to analyze mathematical situations. Asian and African countries also put TG in secondary school mathematics subjects (Gast, 1971; Mashingaidze, 2012; Sinclair & Bruce, 2015; Trimurtini et al., 2021). In Indonesia, TG has mainly been considered in the mathematics curriculum for junior and senior high schools (Badan Standar Kurikulum dan Asesmen Pendidikan, 2022), especially in the KM.

Despite the breadth of TG taught in schools, students need help understanding TG thoroughly, for they are being taught TG directly through a formal approach, leading them to a lack of situational (or informal) interpretation, as reported by several studies. Ada & Kurtuluş (2010) found that students struggle to develop the meaning of translation and rotation due to the lack of geometric meaning. However, there is no problem with the algebraic meaning. Another example is that even students who have passed the ninth grade have unsatisfactory knowledge of basic geometry as they are less elaborate in strategic knowledge about drawing (Rellensmann et al., 2020). Mbusi (2015) also ascertained that students failed to visualize certain transformations and had insufficient opportunity to practice visualization skills during class. The aforementioned examples demonstrated how the formal approach used in early learning of TG has resulted in a shortage of informal, or more precisely, geometric meanings of TG.

Meanwhile, if we are critical of TG from a geometric perspective, the potential learning of TG if linked to the goals of KM, which prioritize creativity, can certainly be explored further. Therefore, to reach that attitude, we need to accommodate the idea that students should develop an understanding of the subject matter from reality to a higher level. One way to do this is by creating experimentally real situations for students, allowing them to visualize a transformation by manipulating a shape and observing the effect of each manipulation (NCTM, 2000).

In another scene, for context, there is a symbol of Indonesian heritage, namely the batik, that contains a blend of geometric shapes and designs, as discussed by Isnanto et al. (2020) and Prahmana and D'Ambrosio (2020). Batik is not just a fabric but also a storyteller of Indonesia's values and history. Furthermore, ethnomathematics studies regarding the batik revealed the phenomenon that it uses the concept of geometry. The batik patterns offer a casual yet meaningful learning environment, especially for Indonesian students, which allows them to interact with geometric concepts (Prahmana & D'Ambrosio, 2020). Therefore, as a medium for students to develop their knowledge of geometrical shapes, they can study transformations accordingly.

This study seizes that opportunity to support students in developing their understanding of TG differently than before. Previously, studies have discussed the learning of transformation geometry using batik, such as by Noerhasmalina and Khasanah (2023) that addressed the idea of all four applications of transformation geometry concept in making the motif of batik Lampung, namely *Siger* motif, *pohon hayat* motif, and *kapal* motif, while Prahmana and D'Ambrosio (2020) of learning geometry transformation using batik Yogyakarta. In this study, we focus our attention on how batik activities can be conducted in a set of activities in which students have the highest possibility of emerging their creativity while maintaining its value. In order to achieve these goals, we need strong foundations based on two considerations: first, to bring a situational context of batik into the classroom in the most effective way possible, and second, to engage students in heuristic approach and learning theories so that a mutually supportive learning process is woven.

As a justification for the first point, we realize that, in conducting batik-making activities, not all procedures can be accommodated as performed by batik craftsmen with complete tools. Therefore, we transform batik-making activities into a digital manipulative environment to optimize technical aspects and tool efficiency. In doing so, we employ digital software, namely CorelDraw, a graphic design software that has transformation features. CorelDraw offers a perspective on batik-making by transitioning the traditional craft into a digital simulation. With its canvas, users can replicate the fabric base of batik, which, for now, we call the 'main pattern.' At the same time, its transformation tools allow for precise design manipulations, simulating one of the specific types of batik creation. By adapting the batik-making process within CorelDraw, we can establish a structured series of tasks that align with both the conventional batik-making steps and the epistemic development of the TG concept. This digital approach offers a more streamlined experience, removing the need for physical equipment and materials and allowing students to concentrate on the essence of transformation concepts while maintaining the appreciation of the value of local wisdom. This fusion of tradition and technology is a remarkable approach promising future evolution in the digital world.

For the second point, in creating such situations that are experimentally real to students, we consider relying on the principles of the domain-specific instructional theory of Realistic Mathematics Education (RME) (Van den Heuvel-Panhuizen, 2020; Gravemeijer, 2020; Gravemeijer & Stephan, 2002; Treffers, 1987; Freudenthal, 1973), in which our mathematics education viewed as a human activity. One of the main principles of RME is guided reinvention, that teachers and assignments should guide students to reinvent mathematics and experience it as a human activity, as we do in this study. Furthermore, to maximize students' learning trajectory in geometry, we use a theory that addresses the development of students' geometrical thinking, developed by van Hiele (Van Hiele, 1986), which the theory was also named after him. According to van Hiele, students pass through five hierarchical levels of reasoning when they learn geometric concepts (levels 0 to 4), even though van Hiele uses only up to level 3 in his research. Later, researchers in the U.S. changed the numbering to range from 1 to 5 (Usiskin, 1982). In this study, we refer to levels 0 to 4: visualization, analysis, abstraction, deduction, and rigor. We use this theory to develop the initial learning trajectory along with the RME theory.

While the discussions about students' difficulty in learning TG continue, the same approaches remain similar over the studies that we mentioned. In joining the discussion, this study aims to develop actionable knowledge of TG in the form of design related to context and yet general enough to use digital manipulative activities in new situations. We propose such knowledge in the form of conjectured Local Instructional Theory (LIT) in the framework of design research, described in the next section.

METHODS

In this study, we sought to develop actionable knowledge that is sensitive to specific contexts and sufficiently general to be applicable across classroom situations within a design research methodological framework. The knowledge acquired in this methodology is articulated either through design principles, conjecture maps, or hypothetical learning trajectories (HLT, Bakker & Smit, 2018; Simon, 1995). This study formulated a conjectured local instruction theory (LIT) throughout the instructional activity. The LIT consists of conjectures regarding the possible learning process and possible ways to support that particular process. Figure 1 illustrates the reflective relation devised by Plomp (2013), which shows that the LIT guides the thinking experiment, while the design, analysis, and instruction experiments also shape the LIT.

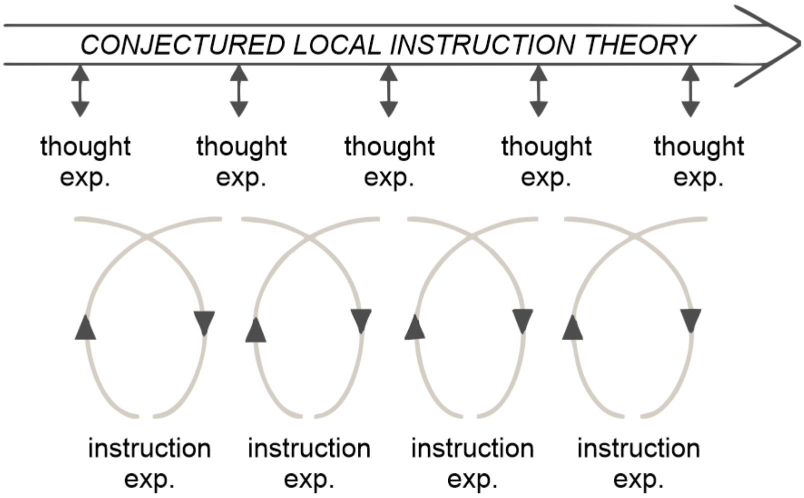


Figure 1. Reflective relationship between theory and experiments (Plomp, 2013)

This study phases began with developing an initial HLT. Simon et al. (2018) described HLT as "the consideration of the learning goal, the learning activities and the thinking and learning the students might engage." Thereby, in this phase we conducted an anticipatory thought experiment on how the proposed instructional activities could be utilised in the classroom and what students might learn when they participate in teaching experiments. We developed the learning activities based on the principles of the domain-specific instructional theory realistic mathematics education (RME, Bakker, 2004; Cobb et al., 2008; Gravemeijer, 2020; Van den Heuvel-panhuizen et al., 2020; Putri et al., 2015; Zulkardi, 2002) and van Hiele's model of development of geometric thought (Crowley, 1987; Usiskin, 1982; Van Hiele, 1986). The activities consist of a pre-assessment test and four teaching experiments.

Next, we sought to establish how the students' actual learning process corresponds to the HLT. We examined the actual condition of whether students' observed learning processes align with what was predicted in the HLT. Therefore, throughout the study, the HLT used as a guide for instructional activities comprises the HLT phases related to conjecturing, enacting, and revising the HLT itself accordingly before we finally formulated a conjectured LIT.

Data Collection

The data presented were collected during the pre-assessment and teaching experiments by way of five 45 min sessions conducted with three single-subjects: Nadeo, Dama, and Subagya (all pseudonyms). The focus is on studying the behaviour of a single individual under certain conditions. They were in their 11th grade in a private school in Surakarta, Indonesia, and demonstrated a level of understanding and capability, yet also represent diverse classroom contexts, to contribute meaningfully to the study. The level of understanding and existing knowledge was gained by our conducted pre-assessment on each of them, outlined in the next section. The pre-assessment data gathered was then used to support a series of two teaching experiments aimed at developing an initial Hypothetical Learning Trajectory (HLT). During these tasks or instructions, interventions were made in the form of interviews with the students, allowing them an opportunity to articulate and reflect their thoughts. Throughout the study, video recordings of the meetings, along with the transcript of interview and students' worksheets analyzed retrospectively in accordance with HLT.

Data Analysis

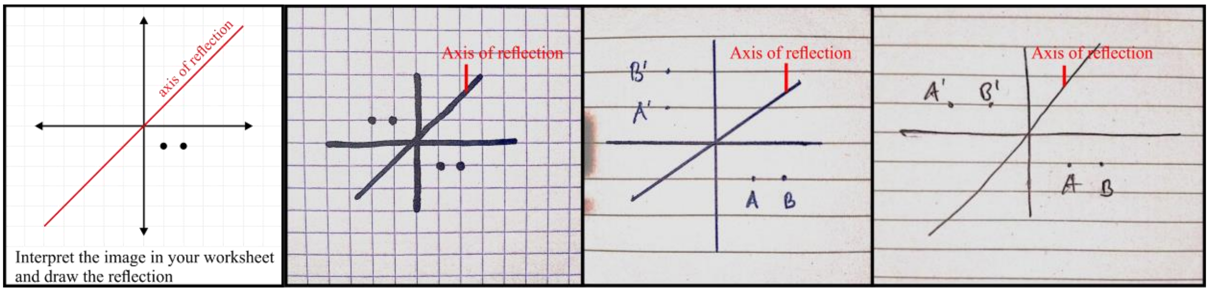
We present and retrospectively analyze the HLT and subjects in parallel order for each activity performed. First, we outline each initial HLT component and then analyze it based on the implementation in the lesson. We examine why and how our design worked and use our findings to evaluate and revise the HLT. In addition, we also analyze the process of the activities carried out by students based according to the aspects of the four geometric thinking. The main results of this study depend not on the design itself but on the underlying principles that explain how and why the design works as the goal of the study is preserved.

RESULTS AND DISCUSSION

This section describes the tasks posed, the results, along with justifications for the tasks that ensued.

Looking Inwards: Informal and formal transformation geometry (Pre-assessment)

Prior to engaging the students in the instructional interventions, we conducted a test for the three students. This test aims to support a series of two teaching experiments aimed at developing the initial HLT. It contained two questions that allow the three students to show their geometrical and algebraic knowledge of transformation. Concerning the first task, we presented a reflection problem (Figure 2a) with two coordinate points and asked the students to determine the reflection of those points in the axis of reflection. As a second task (Figure 3), the students were given a TG problem in a coordinate rotation with a certain angle and direction.



(a) (b) (c) (d)

Figure 2. Students' results from the first pre-assessment task

In the first task, Nadeo (Figure 2b) began to draw the Cartesian plane and two initial dots. Subsequently, he introduced the idea of the "opposite view". He explained how the reflection of an object in a mirror will always be "flipped". After he drew the two dots, he continued drawing two other dots in different positions. The distance from the axis line of the reflection of the corresponding parts of the image and the initial image is the same. The following conversation was part of an interview that presented Nadeo's explanation.

Nadeo : [pointing at the axis of reflection] This is the mirror; the result is the opposite view.
Researcher : What do you mean by "the opposite view?"
Nadeo : These two [pointing at two given dots] are the initial position. I use the characteristics of the objects in the mirror where the image will be "flipped'. So, from my point of view, the images will be on the other side of the two, which means it's both of these [pointing at two drawn dots].

Subagya began marking two points, A and B (Figure 2c). Next, he rotated the paper to make the "mirror" vertical. By changing his point of view, he described the mirror image of the two given points. Another student, Dama (Figure 2d) used the same idea. However, he examined the paper from his right side and saw point A near the mirror, whilst B was further away. Next, he imagined the images to appear the opposite to what he saw. Consequently, he marked two mirror images, with A being the furthest away and B being the closest.

Each student uses their approach to find the mirror image of the first two dots. The students recognised the geometric figures by their shape as "a whole", although they do not know how to obtain the relationship between the starting point and the image. Thus, students at this stage are still at level 2 as regards geometric thinking (Usiskin, 1982).

Find the image A' of the point $A(-2, 3)$ under a rotation by 90° about the origin.

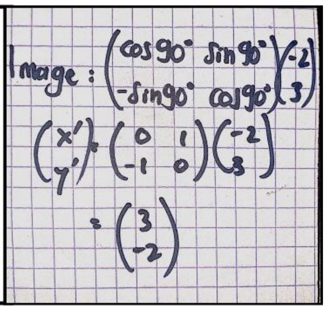


Figure 3. Nadeo's results for the second pre-assessment task

In the second task, the three students responded in similar ways (see Figure 3), as they gave identical responses. We will only discuss Nadeo's answer. He used an algebraic approach to calculate the image of rotation. The rotation matrix he used has the centre (0,0) and an anti-clockwise angle of 90°. As he explained in his conversation with the researcher, he used the formula his teacher had taught him in class to solve this specific problem.

Researcher : Nadeo, why did you use this calculation?
Nadeo : I used the matrix according to the formula.
Researcher : Do you know what kind of matrix that is? Can you explain why that matrix is used?
Nadeo : N… no, I think I need help.

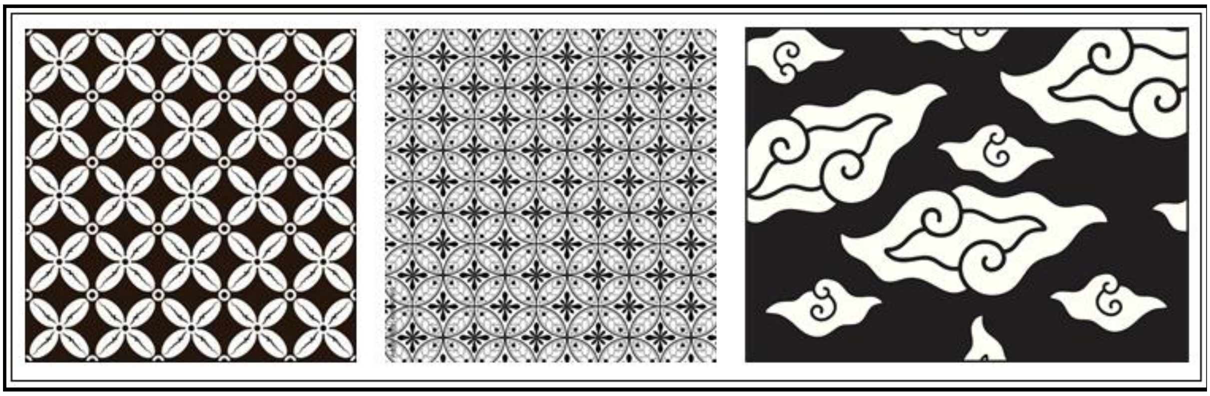
Concerning the second task, the student could solve the problem mechanically. However, he was not able to justify their strategy by means of TG problems in a geometric situation (first question). He attempted to solve the problem using his common sense. Overall, all the students were able to complete Activity 1 algebraically.

The results obtained from the pre-assessment tasks on informal and formal transformation geometry provide insights in developing the initial HLT. The diversity in approaches and thought processes exhibited by the students, such as Nadeo's concept of the "opposite view" and Subagya's and Dama's unique perspectives on mirror imaging, highlight students' varied cognitive paths in geometric thinking. These diverse approaches and the obstacles encountered, such as the students' ability to solve problems mechanically but their struggle to justify their strategies, underscore the need for a learning trajectory that is attuned to the multifaceted nature of student thinking and learning. The students' reliance on algebraic methods and their recognition of geometric figures as "a whole" without understanding the underlying relationships between starting points and images reveal potential areas of focus and conjectures that might arise in the learning trajectory. In essence, the results from the pre-assessment tasks will serve as a foundation in constructing a learning trajectory that is responsive to the diverse and complex landscape of student understanding and learning in transformation geometry. The HLT, along with the result and analysis, starts from the following sections.

In the Beauty of Batik: Main Pattern (Activity 1)

This activity begins with building the concept of transformation to see repeating patterns using *batik* interpretation as an introduction. As stated by van Hiele, students use visual perception and recognise geometric figures by their shape as "a whole". Thus, we designed Activity 2 to make students realise that a whole *batik* consists of patterns arranged by certain rules. By exploring those rules, we expect that students begin to discuss the "movement" of an object and the creation of *batik*. We will use the term "main pattern" for the part of the *batik* that students use as the moving object to create *batik*. We emphasise that the main pattern is not the smallest part of the image (*batik*), but a certain part of the *batik* which, if the collection of the main patterns is arranged, can become a whole *batik.*

Firstly, we asked students to determine the main pattern of three different *batiks*. Students were expected to informally analyse the properties of transformation, analyse its relationships, create meaningful definitions, as well as justify their reasoning as van Hiele describes Level 1 of the Geometric Thinking approach.



(a) (b) (c)

Figure 4. *Batiks* used in Activity 1

We anticipate that students will begin to identify a *batik* pattern and notice the position formed by its components. We also expect that students will focus on the stunning visual images. For example, students will look at the shape of a "flower" in Figure 4a and make it the main pattern as they see the 'floral pattern' repeating itself. They might also attempt to divide the 'floral pattern' into smaller shapes. From the pattern, we hope that students acquire their own ideas for the main pattern and articulate their thoughts.

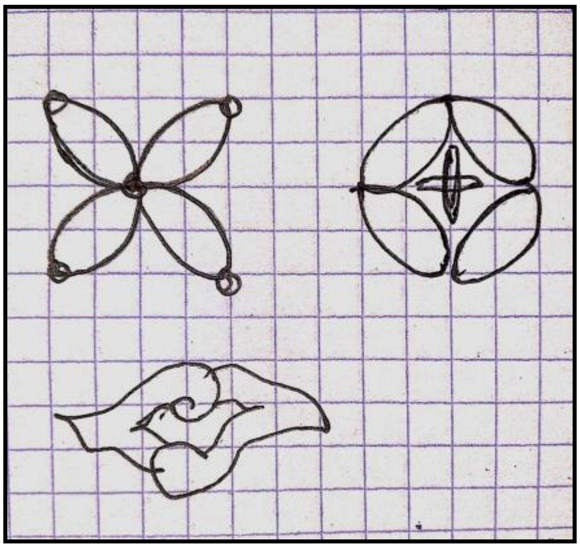


Figure 5. The result of Nadeo completing Activity 1

Regarding this task, each of the students quickly conveyed their ideas in relation to the main pattern. Nadeo looked at the first *batik* (Figure 4a) and immediately drew a flower shape (Figure 5). Within a few minutes, he drew the petals and then continued by adding in four small circles at the end of the petals, as illustrated in Figure 5. He explained the flower shape as it was 'repeatedly' visible in the *batik*. He concluded that if he already had a picture, he could reverse the step by simply moving that picture. He made a less obvious step when drawing the third pattern (Figure 4c) by just looking at it for a moment. He then continued by drawing a "cloud", as shown in Figure 6a. After he had completed his drawing, we asked how his main pattern would allow him to recreate the whole *batik*. With this particular question, we sought to move toward the concept of movement and duplication. The following discussion is part of an interview with Nadeo regarding his answer.

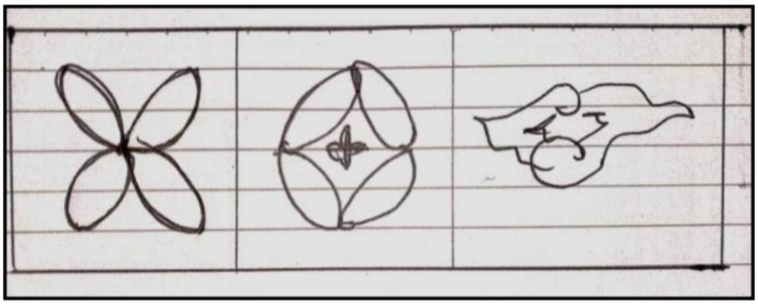
Researcher : Let's call this image you have the 'main pattern'. With this, how will you form the initial *batik* composition?

Nadeo : The image is… repeated, and because this one [showing his main pattern] is repeated by moving it to the right and by moving it downwards.

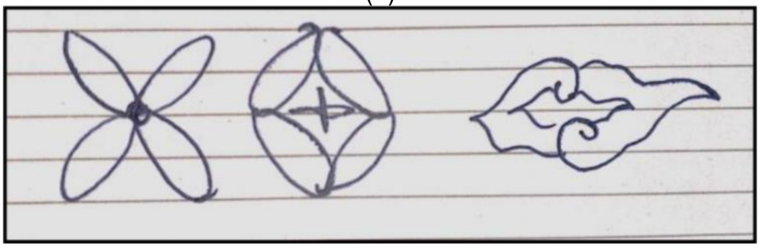
Researcher : If you shift it, doesn't it just move? I thought your *batik* would be full of patterns.

Nadeo : [Thinking carefully]. Well it's actually also duplicated. In the process of making the *batik*, it uses a stamp and the results are like that.

In the conversation above, the term "moving", that Nadeo uses, was clarified with a follow-up question, emphasising the actual concept of transformation. He understood what duplication meant in the context of *batik*. He immediately replied with "just a movement" and indicated that what he was doing was shifting and not duplicating. Afterwards, when he was ready to create a whole *batik* image that was the goal of Activity 2, he applied a combination of duplicate and transformation. Students provide arguments to justify their reasoning at the abstraction level. He also used grid paper to illustrate his main pattern. Looking at the following statement, he described the transformation as "the way we move the object". His experience in learning TG on this occasion has attained Level 2 (abstraction) of van Hiele's theory.



(a)



(b)

Figure 6. Subagya and Dama' result for Activity 2

Figures 6a and 6b show the main patterns created by Subagya and Dama in which two images look very similar. Firstly, Subagya drew a square to divide three *batiks* into three main patterns. Next, he drew a "floral pattern", as Subagya did in Figure 6a. The second and third main patterns interpreted by Nadeo, Subagya, and Dama did not significantly differ. Three of them make four ovals with flowers and one cloud.

Subagya continued by answering questions about how to produce the initial *batik* composition using the main pattern. The idea that he conveyed was a "*batik* stamp" in which *batik* would be arranged according to the desired pattern.

Researcher : Let's call this image you have the 'main pattern.' With this, how will you form the initial *batik* composition?

Subagya : Mmmm…, that's possible… I will print it like a *batik* craftsman makes *batik* with a stamp, so the stamp is shifted like that.

Meanwhile, Dama decided to "place the main pattern over and over again", as demonstrated in the following interview:

Researcher : Let's call this image you have the 'main pattern.' With this, how will you form the initial *batik* composition?

Dama : Well, because I saw the *batik* composition repeated, I will just "place the main pattern over and over again".

Researcher : How do you "put the main pattern over and over again"? Are there any special rules?

Dama : Of course, maybe related to the position, because *batik* is well arranged.

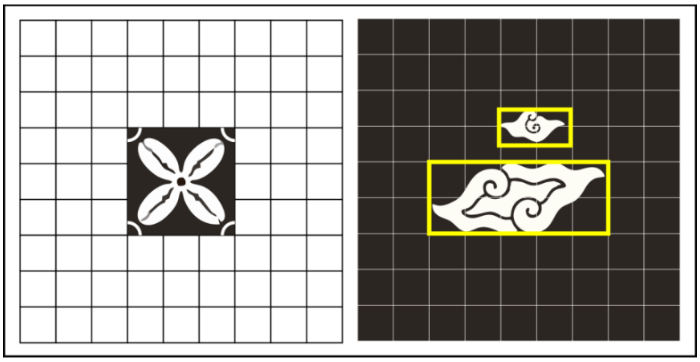
During Activity 1, students have shown that they have developed ideas and strategies to determine the main pattern. Moreover, they also understand that the main pattern is used, specifically in this context, as the basis of the overall *batik* pattern. Furthermore, they also explore the concept of transformation in batik patterns, reflecting the visual phase of van Hiele's geometric thinking model where students recognize geometric figures as a “whole.” The next activity will use the pattern as a material for student investigations in relation to transforming a visual geometric object. Thus, students begin by understanding how each component of the transformation works, then use those ideas in transformations in mathematics. This activity not only helps students develop ideas and strategies to determine the main pattern but also enables them to understand the transformation of visual geometric objects.

They Move Elegantly: Introduction to Transformation Geometry (Activity 2)

We designed Activity 2 to introduce the types and characteristics of various transformations in TG, namely translation, rotation, reflection, and dilation. We invite students to use the types of transformation shown in the main pattern and use the rules that are important to *batik*, as students use their informal knowledge in formal deduction (Ngirishi & Bansilal, 2019). Thus, the idea of Activity 2 is the process of re-unitizing the main pattern obtained in Activity 2 into a complete *batik* composition using CorelDraw® software (from now on, referred to as Corel). Students can use the main pattern as a model. Moreover, they use their strategies regarding the situation given in the problem. As an introduction, the main patterns mentioned by our students were narrowed down to two types, including *batik* 1 and 3. Furthermore, the patterns made by students were previously interpreted digitally in Corel, along with the initial *batik* images (Figure 7).

The initial trajectory regarding developing the TG concept is when students use each transformation in Corel. In the process, they can realize the function of each transformation used. When students discover what properties are available in transformations (i.e., translations with shifts of x units and y units), they will realize that these properties will be used in solving TG problems).

Activity 3 allows students to explore transformation as deeply and creatively as possible. We anticipate that the student would begin by trying every feature provided. In addition, they may realise that making *batik* compositions requires transformation and duplication for the first time. In this stage, students will understand that transformation differs from transformation and duplication. Main pattern 1 (Figure 7a) is designed to be simpler than main pattern 2 (Figure 7b) predicting that students will understand transformations more easily with limitations on left-right and up-down shifts. In main pattern 2, we recognise that students could not produce the same *batik* shown first because of its complexity. However, we hope that students can use the whole transformation and understand what they are doing. The final goal is to ensure that each student can achieve geometric thinking skills Level 3 (Usiskin, 1982), in which students will think informally by way of applying visual *batik* in their activities.



(a) (b)

Figure 7. Main pattern set to Corel.

Students use transformation and duplication to develop the main pattern into a *batik* composition consisting of the following Corel features, namely 'translate,' 'rotate' and 'scale and mirror.' Each component can be used individually according to its function in the TG with each available property (Figure 8). In addition, Corel also provides a 'Copies' feature' which is located under the transformation and indicates the desired number of duplications in three previous parts.

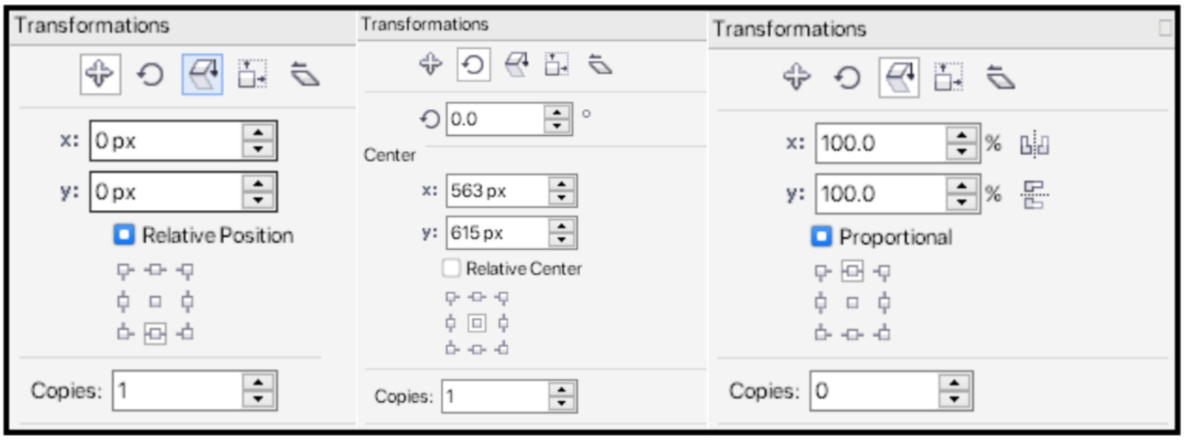


Figure 8. Transformation feature in CorelDraw®

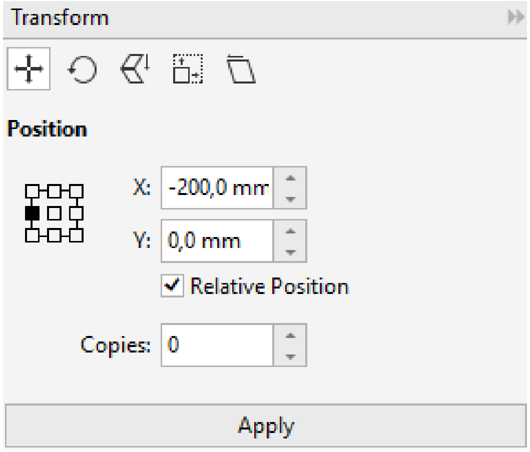
In Activity 3, the process starts when Subagya receives the first pattern from Figure 3. He attempted all the transformation features in Corel by entering input and clicking the apply button. The following interview shows what Subagya did regarding this process:

Subagya : [Puts input into translation, clicks 'apply' button]
Subagya : How can I change what I have done?
Researcher : You can press Ctrl+Z to undo.
Subagya : [using the same steps on rotate, scale and reflect].
Subagya : Ok, I got it.
Researcher : What did you get earlier? What would you say to your friends?
Subagya : I know what to do with translation. The inputs are x and y, rotation with angle, scale with percentage and reflect based on horizontal and vertical.

Then Subagya continued the process. However, he was confused when he was about to *create* a *batik,* as he realised that he was only transforming.

Subagya : It's only moving. How do I duplicate this object?
Researcher : Why do you want to duplicate? Don't you want to transform?
Subagya : Of course, but we must also duplicate the main pattern to make a *batik* composition.
Researcher : You can use the 'Copies' feature.

Subagya immediately understood that he had to enter a value of 200px, which is the size of the main pattern. As he shifted 200px and chose the 'middle left' direction, the Corel input automatically changed to -200 as it shows the translation direction. Subsequently, the image is translated, resulting in the image being right to the left of the initial object. He continued until he created the whole *batik,* as displayed in Figure 8. In the end, Subagya made the entire Main Pattern 1 *batik* with minimal difficulties in understanding the TG concept.



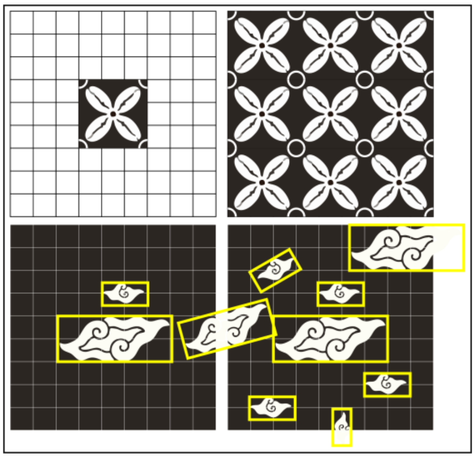


Figure 9. Subagya's process and result for Activity 3

In *batik* 2 (Figure 4c), there are two types of main patterns, namely large and small "clouds". Subagya continued what he learned in main pattern 1 with all the transformation properties being used, including translation, rotation, reflection and scale. First, he translates by -353px on the x-axis, rotates it 15° anti-clockwise and performs 80% scaling. However, he translated 400px on the y-axis and 300px on the x-axis with the big cloud. Then, he changed the initial object to a small cloud. He translated -400px on the x-axis and rotated it 90° anti-clockwise. Afterwards, he transforms the small start three times with each -300px on the y-axis and 100px on the x-axis, -350px on the y-axis and -150px on the x-axis. This is followed by 50px on the y-axis and -150px x-axis reflection against the y-axis then a 30° rotation. The final result is also revealed in Figure 9.

The other students, Nadeo and Dama, went through a more complicated process in completing Activity 3. When they attempted to use the transform feature, they began to recognise its properties. However, both experienced problems applying the transformation to the main pattern. In main pattern 1, both of them needed help understanding the translation value in the Corel. They used a 'guess and check' approach, entering numerous values to achieve the correct position. They changed this approach when they realised that translation should be related to the size of the main pattern.

Dama : [Silent for a while, translates main pattern 1].
Researcher : Do you think the number is correct?
Dama : Wait, what shift values should be entered? I translated this, but it doesn't move right next to it.
Researcher : Well, you can see the size of the main pattern at the top left of the screen.
Dama : Ok, and… uh, I got it. So, it should be as big as the main pattern.
Researcher : Why?
Dama : If we make the same size [executing the task] … here it is [showing the result].

In the second challenge (Figure 9), Dama began with a translation of 400px on the y-axis followed by 250px on the x-axis. Returning to the first object, he continues with a translation of -250px on the x-axis, then rotates it 15°. Switching to the “little cloud,” he rotates the cloud by 90° anti-clockwise and translates 500px on the y-axis. Then, once he went back to the first small cloud, Dama translates -300px and rotates it 30° anti-clockwise (Figure 10).

In this situation, Subagya and Dama tried the available transformation features one at a time. Dama experienced no significant difficulties. He tried the existing features one by one and immediately knew the various types of transformations and their behaviour. He also understood each input from the transformation and the transformation's results. However, he was not able to determine the amount required to transform the main pattern correctly. In addition, he also made an error when he translated to left-down. However, he immediately realised and replaced the input value with a negative number. In main pattern 2, Dama does a similar thing to Subagya, as Figure 10b explains.

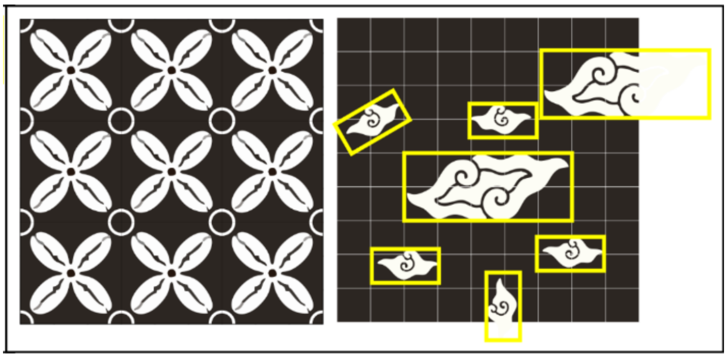


Figure 10. Nadeo’s result for Activity 3

In this activity, we have observed students using transformation in translation, rotation, reflection, and dilation. In the case of simple transformations, such as in Figures 4a and 4b, students will perform the transformation precisely because the existing patterns can be quickly produced. However, in the case of the *batik* in Figure 4c, the students could not transform strictly at the initial image position.

The students’ journey through the activity reveals the development of an understanding of transformation geometry concepts. Subagya explores the transformation features in Corel, initially struggling with the concept of duplication but eventually grasping the transformation and duplication processes to create batik compositions. Other students, like Nadeo and Dama, experience challenges in applying transformations, resorting to a ‘guess and check’ approach before understanding the relation of translation to the size of the main pattern. Despite the challenges, the students manage to understand the various types of transformations and their applications, moving from visual objects to a more abstract understanding of transformations. This progression aligns with the van Hiele model, illustrating a transition from visual to analytical geometric thinking, allowing a more formal and abstract understanding of geometric concepts. With this principle in mind, in the next activity, we will take students to a more mathematical world by inviting them to change the transformed object to a more abstract object (coordinates), allowing them to move from an informal to a more formal understanding of transformation.

Transition from Visualized Batik into Coordinates Abstraction (Activity 3)

The following activity invites students to use a more formal form of TG while creating a *batik*. The task aims to make students see their problems in the world of mathematics, it uses them to solve and see the results in context. The next problem will be modified into a more formal form. In this case, we develop students' abilities vertically (Toole, 2001). Students use maths-related strategies to solve the TG problems, attain the formal level, as well as apply conventional procedures and notations. This bridge contributes to the development of the abstraction of transforming the objects more mathematically.

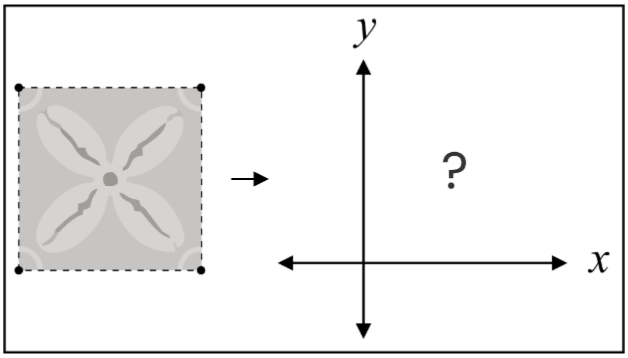


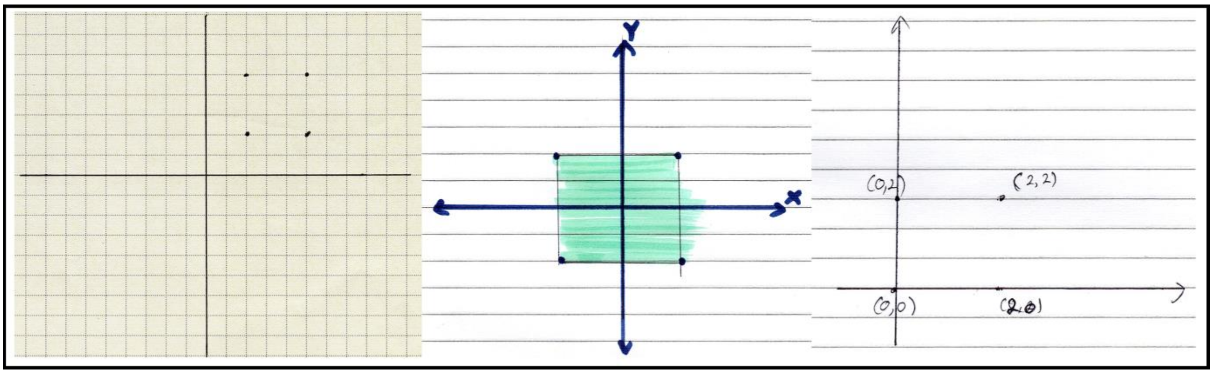
Figure 11. Task for Activity 4

We asked the students to think and interpret their main pattern in Cartesian coordinates (Figure 11). We also reduce the object to only one type of main pattern 1. We anticipate that students may describe a square, four coordinates, or only one coordinate point**.** However, we conducted interviews to ensure that students no longer saw what they transformed as visual objects but the composition of the points in Cartesian coordinates. Their interpretation of the results is illustrated in Figure 12.

Three students started this activity by drawing two axes and drawing four points. However, each student places the four points at a different location. Dama (Figure 12a) only drew four dots, while Subagya made a more varied visualization (Figure 12b) by adding a dash and green as a marker of its area. Nadeo wrote the Cartesian coordinates in his drawing (Figure 12c).

Our intervention pertains to the projection of the coordinates they use. Hence, we conducted interviews and asked them:

Researcher : You made four dots. What does it represent?
Dama : The shape of the box is made of *batik* because of the Cartesian field, so I created a point.
Researcher : Ok, since you made points in Cartesian, do the dots show any specific coordinates?
Dama : I think so.
Researcher : And what are those coordinates?
Dama : Those are…, $(2{,}6)$, $(6{,}6)$, $(2{,}2)$, $(6{,}2)$ (*see Figure 12a*).



(a) (b) (c)

Figure 12. The result of Activity 4 for Dama (a), Subagya (b), Nadeo (b)

Researcher : You made four dots. What does it represent?
Subagya : I made it in the middle. I think all four of them have coordinates.
Researcher : And what are those coordinates?
Subagya : $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$.
Researcher : Why did you translate the *batik* representation into four coordinates?
Nadeo : Well, I think if we look at the Cartesian field, then we are talking about coordinate points.

During this activity, students are transitioning from a visual representation of batik to a more abstract, coordinate-based representation, which progresses from visual to more abstract and formal deductive reasoning. The students' varied responses and representations during the activity illustrate their evolving understanding and application of mathematical concepts and coordinates. For instance, some students represented the batik visually with dots, while others incorporated Cartesian coordinates, indicating a move towards formal deduction, the third level in the van Hiele model (Usiskin, 1982; Van Hiele, 1986). This progression is crucial as it allows students to analyze and transform sets of coordinates and visualize them as geometric shapes, such as squares, for a better understanding of transformation concepts.

The activity thus serves as a bridge to more advanced mathematical problems and formal tasks. It is crucial to preserve the elements that facilitate the transition from visual to abstract thinking, as per the van Hiele model. However, continuous feedback mechanisms should be strengthened to monitor students' progress and to provide timely support where needed, ensuring a smoother transition from informal to formal mathematical thinking.

Climbing to the Top of the Iceberg: General and Formal Activity (Activity 4)

The following task is designed to complete the abstraction process. We replaced *batik* with the coordinates of the transition activity. Each student uses their own coordinates they had in the previous activity. In the following tasks, students had to use the concepts they learned earlier. We anticipate that students will undertake the transformation according to what they did in Corel. They may well experience some difficulty using notation (i.e., translational notation with a matrix). We assist them with the tasks by way of discussions or interviews. In this case, students will not use matrices because they have not been given the terminology or explanations about transformations using matrices. Thus, they might only use the interpretation from Corel to obtain the results of their object transformation and include the coordinates based on the image, which is where they recognise the transformation via their own understanding.

The goal is to develop students' understanding regarding the transformation procedure, to use their abstraction and notation, and to reinvent the concept of TG. The following tasks describe how students reshape their ideas formally (see Figure 13).

Using the coordinates you have made, perform the following transformations:

1. The value of translation you used in Corel was a and b. Now we will use different terms, namely x and y, written in the coordinate (x, y). Find the coordinate of image A' If a=4 and b=4.

   Note: you can illustrate or write down the coordinates of the image you produce.
2. Now you have image A'. As we have done in Corel, now perform a rotation by 90° anti-clockwise with the object's centre as the anchor. Then, write down all the new images B'.
3. Find image C' of the B' under a reflection with the axis of reflection is the vertical line through the centre of your object.
4. Perform a dilation and find the image D' by the scalar 2 (200%) about the origin.

Figure 13. Tasks for Activity 4

In this task (see Figure 13), the students are asked to use the coordinates they created in Activity 4. Here, Dama does not encounter any problems.

Researcher : What did you get for the values of a and b?
Dama : It's like in x and y in Corel.
Researcher : So, what's the difference with the coordinates (x, y)?
Dama : I have four coordinates (x, y) here. I add each one, see this [shows the process of adding it].
Dama : B… but… in this reflection process, I think the final image will remain the same, even though the image is different.
Researcher : The image is the same and it is different. That is intriguing. Help me understand what you mean.
Dama : Because the coordinate position is "on the other side", it is like, I mean, the box is flipped, so these coordinates are right here [indicating the coordinates of one moving to the other side].

Afterwards, Dama is confused whilst interpreting his reflection process. He encounters some difficulty in determining the midpoint of all the coordinates and he is just reflecting by "imagining" the four coordinates are *batik* patterns, as he said in the interview.

Researcher : How do you reflect on that object? Where do the coordinates come from?
Dama : I'm confused here. In Corel, I only have a horizontal reflection or vertical reflection, so I did that.
Researcher : What line do you reflect on?
Dama : I assume the coordinates $(1,1)$ are the midpoint, so I reflect on the vertical and horizontal lines. I imagine this is the previous main pattern.
Researcher : Then the dilation?
Dama : Yes, here I multiply each coordinate by 2 because of 200%.
Researcher : Which coordinates did you mean?
Dama : After the reflection earlier.

In Activity 5, Dama has already started his abstraction step, shown by taking the centre coordinate and 200% magnification. He perceives the relationships between properties and figures. Furthermore, he developed a meaningful process and was able to give arguments to justify his reasoning. In this condition, he has attained the level of abstraction and deduction (Usiskin, 1982; Vojkuvkova, 2012).

At this moment, Subagya and Nadeo went through a different process. Subagya had trouble with task number 1 because he could not differentiate the notations a and b in task 1 and x and y in Corel. He understood x and y as translational quantities that should be input into the system (Corel) as revealed in the following interview:

Subagya : [pause] …, I don't get it, how do I input the x and y values?
Researcher : Well, you used to do it when you used Corel.
Subagya : No, it's different. In Corel I input the x and y values, here x and y are my coordinates.
Researcher : Then what are a and b?
Subagya : …., Uh I see.

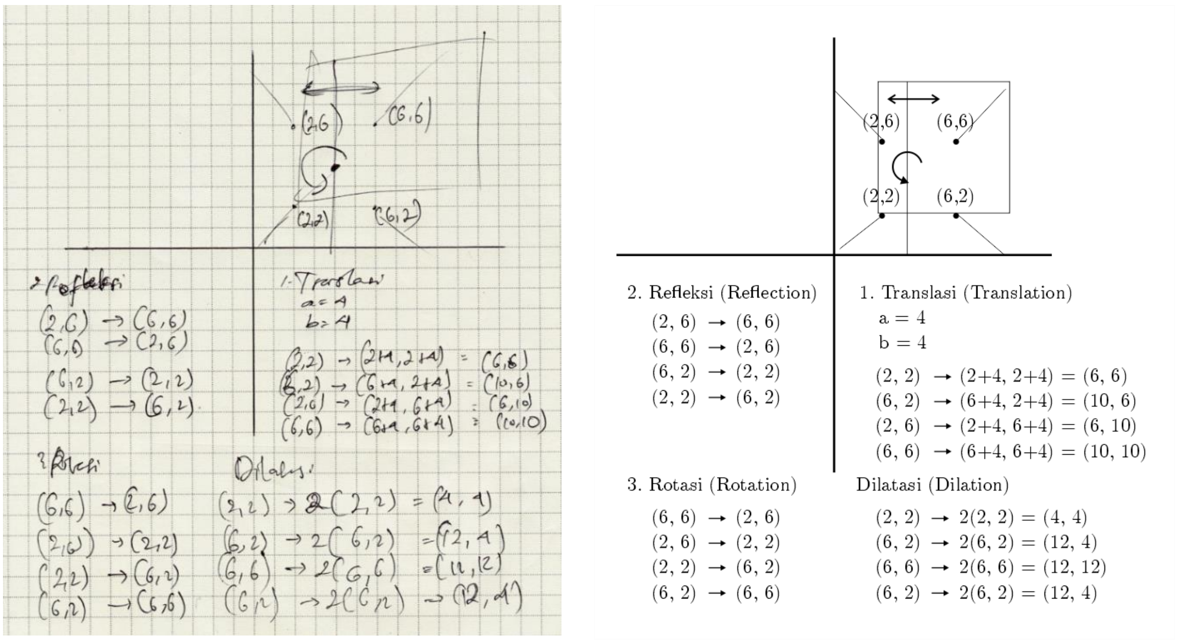


Figure 14. Dama’s result for Activity 4

However, he chose not to redraw the result of the transformation, but only to mention the coordinates and mark it out (Figure 14).

Subagya : In question 1, because all are added by 4, the result is $(1+4,1+4)=(5,5)$; $((-1)+4,1+4)=(3,5)$; $(-1+4,-1+4)=(3,3)$ and $(1+4,-1+4)=(5,3)$. Then the result of the rotation at $90°$ is the same image. And when each one is scaled to 200% it will be $(10,10)$, $(6,10)$, $(6,6)$ and $(10,6)$.
Researcher : Let’s see if I understand what you said. I hear you say that the result of the rotation at 90 degrees and the reflection is the same. Are we sure that the vertices of your new object have the exact same coordinates as your old object?
Subagya : N… no, I mean, the picture is the same because it is only rotated and reflected.
Researcher : So, what happens to the vertices?

Subagya : Well, wait [reviewing his work]. When we take example 1, for instance, at coordinates $(5, 5)$, when it is rotated 90 degrees it will move to $(3,3)$. It moves like that, everything else is the same. Reflection has no difference. Here, I reflect vertically so that the coordinates $(5,5)$ become $(3,3)$ and vice versa.

The main problem experienced by Subagya was the difference in the notation between Corel and what he had learned from his teacher. In Corel, the inputted translation values are in *x* and *y* terms, but in this case, we use *a* and *b* instead. We use a and b to distinguish them from the coordinate notation formed. In addition, Subagya still understands the results of reflection and rotation as a visualisation of the boxes and the coordinates as in translation. However, in his drawing (Figure 14) Subagya demonstrates that he understands the image's coordinates.

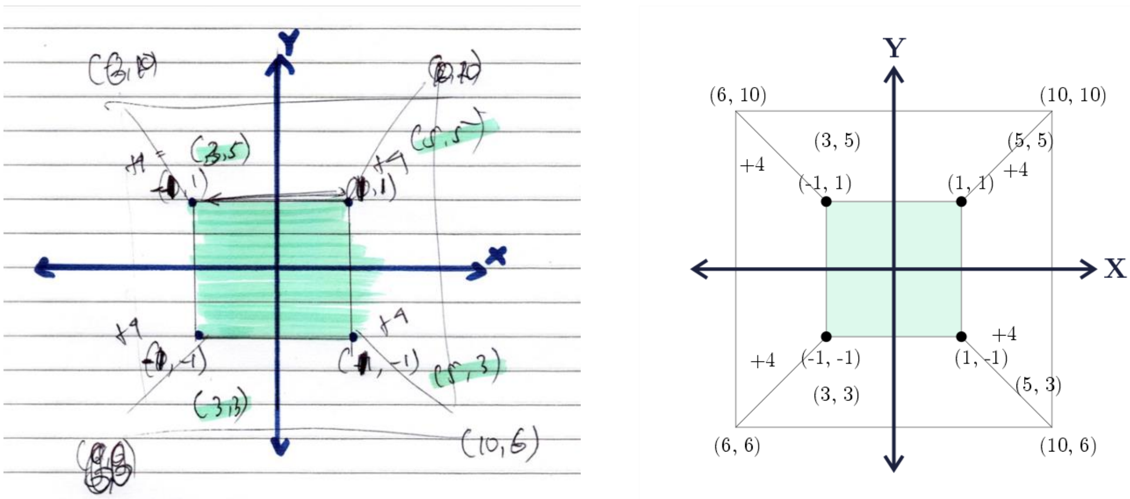


Figure 15. Subagya's result for Activity 4

Nadeo had a similar problem. In problem number 1, he still uses x and y as the quantities he uses in translating the coordinates. However, we first proceeded to guide Nadeo through the x and y values.

Figure 15 shows how Nadeo wrote "$0 + x$" and "$0 + y$," which became the notation for adding or translating coordinates. He continued by inputting x and y with a value of 4 which should be the value of a and b. In addition, he also miscalculated the dilation which resulted in him being confused about the resulting coordinates.

Researcher : Hi Nadeo, I see you added x and y. Help me understand that.
Nadeo : Because that's what I do in Corel.
Researcher : Isn't the problem a and b?
Nadeo : …., Yes, maybe I will replace it.

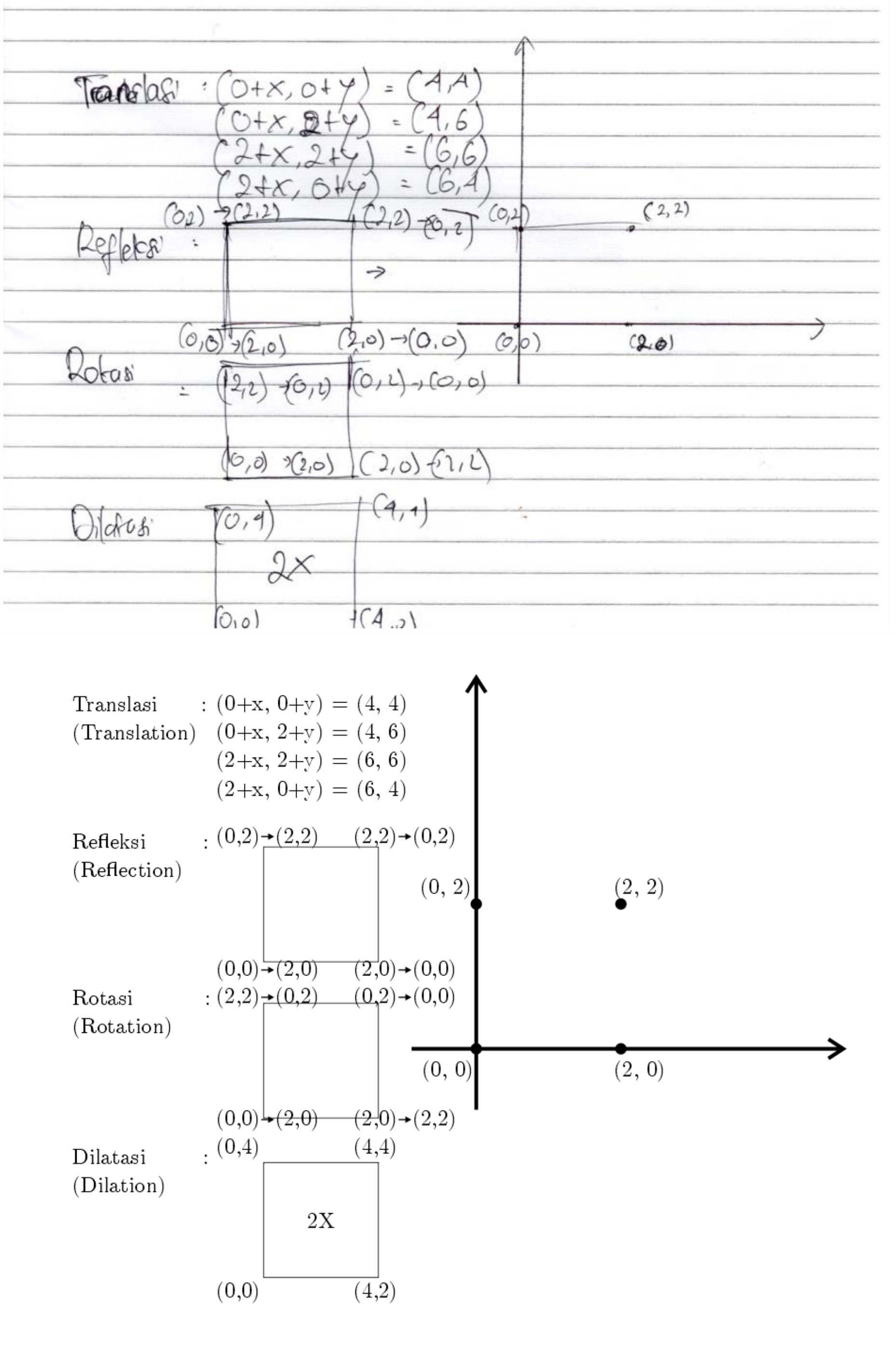


Figure 16. Nadeo's result for Activity 5

After completing the dilation, Nadeo grasped how what he was doing in Activity 5 (see Figure 16) was related to what he was doing in the activity in Corel (Activity 3). He still used the squares he had made and put the coordinates on all four sides. He grasped that he had to transform the four coordinates to obtain the image and that these four coordinates formed the "other square" resulting from the transformation process. However, he may not fully understand the transformation process as he is still stuck with the notation he uses.

In the context of formal level or rigor, the students in the described activities navigate abstract processes, explore transformations, and grapple with notations and coordinates. This level involves logical reasoning and deduction, requiring students to understand properties and relationships within geometric figures and to construct proofs (Usiskin, 1982; Van Hiele, 1986). Dama, for instance, is seen to be operating at this level as he abstracts and perceives relationships between properties and figures, developing meaningful processes and providing arguments to justify his reasoning. He can abstract and magnify the center coordinates by 200%, demonstrating an understanding of transformation procedures and the relationships between different geometric properties, indicative of van Hiele's formal level of geometric thinking.

However, the transition to this formal level is marked by challenges, particularly in notation and abstraction. Subagya and Nadeo struggle with transitioning from the concrete notations used in Corel to the abstract notations used in their tasks, indicative of the difficulties often encountered at this level of geometric thinking. Subagya, for example, struggles with differentiating between the notations used in Corel and those taught by the teacher, demonstrating difficulty in abstracting from the concrete to the formal, a key component of van Hiele's formal level. He visualizes the results of reflection and rotation as visual representations of boxes and coordinates. Still, his drawing demonstrates an understanding of the image's coordinates, showing a glimpse of formal geometric thinking.

In reflecting on the learning trajectory, it is crucial to preserve the structured approach to learning, which allows students to gradually progress through the levels of geometric thinking, from concrete to abstract, as outlined by van Hiele. There are aspects that require refinement to enhance the learning trajectory. The transition from concrete to abstract, particularly in notation, must be smoother to prevent confusion, as seen with Subagya and Nadeo. A more gradual introduction to abstract notations and concepts, with clear links to their concrete counterparts, could aid in this transition. Balancing preservation and change in this manner can optimize the learning trajectory, ensuring that students can successfully navigate the complexities of formal geometric thinking and finally helping students to rigorously develop their knowledge of transformation geometry.

CONCLUSION

Considering how mathematics education is implemented in Indonesia, to achieve the development of an appreciation of the usefulness of mathematics in daily life (curiosity, attention, and interest in learning mathematics), learning through a direct formal approach, including learning the concept of TG, will not assist students understanding. Therefore, we recommend that teachers integrate and develop this design in learning activities relating to TG with students in the eleventh grade. The design principles and LIT must be tested in various contexts to ensure consistent results and potentially uncover new conjectures.

Recalling this study aims to investigate the effects of learning activities. The analysis of the data described above and the justification of the processes during the teaching experiment indicate a compelling trajectory for students learning transformation geometry. The design focuses on students' active engagement with batik as a context for learning and its association with concepts in transformation geometry. In classroom learning, using *batik* contexts enables students to imagine situations and reinvent the concept of geometric transformation. Moreover, activities that are integrated with the context and principles of the RME approach allow students to develop an ordinal understanding of TG.

In conclusion, activities designed in such a way resulted in students developing their knowledge of TG. The results indicate that students can learn according to the designed learning trajectory. Likewise, the situation that occurs in this study may well be different depending on students' initial abilities. However, to note the limitations of this research, the focus on a specific single subject has to put in the consideration, which means the findings might need to be more generalizable to a broader context.

Nevertheless, the study was conducted under the design research framework, which, while offering depth and specificity, might not capture the nuances of diverse learning environments or varied student populations. Given these constraints, future experiments are recommended.

A Local Instruction Theory on Teaching Transformation Geometry in Eleventh Grade

A significant result in this design research is the local instructional theory (LIT) on learning TG in the eleventh grade. In their study, Gravemeijer and Cobb (2006) state that LIT describes the conjecture of learning activity on a particular topic. Furthermore, this study offers a plan that teachers can use as guidance to choose learning activities and compose an HLT.

Our description of LIT is based on the conjectures from the teaching experiment. When students' understanding of TG has not been formed, either geometrically or algebraically, using *batik* as a context can be an effective alternative method to develop students' understanding of their imagination of TG. This is based on the conjecture relating to a particular definition of the main pattern where students can reinvent their understanding of the characteristics of an object that is transformed in specific rules through patterns formed in *batik*. Through the main pattern idea and the recognized characteristics of geometric patterns, students triggered their curiosity to learn more about the TG concept in a formal way. The teaching method involves providing a pattern for students to use as a starting point. The students then use transformation, acting to the pattern and resulting in a complete batik design. Students can understand the characteristics of each transformation, namely translation, rotation, reflection, and dilation, using *batik* as an informal context. This is based on conjectures that the breadth and depth of use of the overall transformation features in Corel resulted from the process of making a whole *batik* with all of the transformation properties. Regarding the transition, the main pattern is interpreted into Cartesian coordinates, which prompts students to use ideas and processes similar to what they applied in Corel. Thus, students can practice understanding and solving simple formal TG problems and then understand them with further notation and formulas.

Acknowledgments

This study was made possible thanks to researchers and students' collaborative efforts in the 2021 International Supervision and Collaboration Program of Universitas Sebelas Maret. We would also like to acknowledge the invaluable contributions of the classes and teachers.

Declarations

Author Contribution	:	SS: Conceptualization, Writing - Original Draft, Editing, and Visualization. MD: Writing - Original Draft, Formal analysis, and Methodology. AH: Methodology, Project administration, and Funding acquisition. TAK: Writing - Review & Editing and Validation. LF: Writing - Review & Editing, Validation and Supervision.
Funding Statement	:	This research was conducted with the support of independent funding sources. The views expressed in this study are those of the authors and do not necessarily reflect the views of the funding organizations.

Conflict of Interest : The authors declare no conflict of interest.
Additional Information : Additional information is available for this paper.

REFERENCES

Ada, T., & Kurtuluş, A. (2010). Students' misconceptions and errors in transformation geometry. *International Journal of Mathematical Education in Science and Technology*, *41*(7), 901-909. https://doi.org/10.1080/0020739X.2010.486451

Badan Standar Kurikulum dan Asesmen Pendidikan. (2022). Capaian Pembelajaran Mata Pelajaran Matematika Fase A - Fase F. *Kementrian Pendidikan Dan Kebudayaan Riset Dan Teknologi Republik Indonesia*, 1–36. https://kurikulum.kemdikbud.go.id/kurikulum-merdeka/capaian-pembelajaran

Bakker, A. (2004). Design research in statistics education: On symbolizing and computer tools. In *Igitur-Archive.Library.Uu.Nl*. http://igitur-archive.library.uu.nl/dissertations/2004-0513-153943/UUindex.html

Bakker, A., & Smit, J. (2018). Design Research in Education (pp. 255–271). Routledge. https://doi.org/10.4324/9780203701010

Cobb, P., Zhao, Q., & Visnovska, J. (2008). Learning from and Adapting the Theory of Realistic Mathematics education. *Éducation Et Didactique*, *2–1*, 105–124. https://doi.org/10.4000/educationdidactique.276

Crowley, M. (1987). The van Hiele model of the development of geometric thought. *Learning and Teaching Geometry, K-12*, 1–16. http://www.csmate.colostate.edu/docs/math/mathactivities/june2007/The van Hiele Model of the Development of Geometric Thought.pdf

Depdiknas. (2006). *Permendiknas No 22 Tahun 2006 Tentang Standar Isi*. Jakarta: Depdiknas.

Evbuomwan, D. (2013). *An investigation into the difficulties faced by Form C students in the learning of transformation geometry in Lesotho secondary schools*. Doctoral Dissertation. South Africa: University of South Africa.

Freudenthal, H. (1973). *The Case of Geometry. In: Mathematics as an Educational Task*. Dordrecht: Springer. https://doi.org/10.1007/978-94-010-2903-2_16

Gast, R. H. (1971). The High School Geometry Controversy: Is Transformation Geometry the Answer? *The Mathematics Teacher*, *64*(1), 37–40. https://doi.org/10.5951/MT.64.1.0037

Gravemeijer, K. (2020). Emergent Modeling: an RME Design Heuristic Elaborated in a Series of Examples. *Journal of the International Society for Design and Development in Education*, *4*(13), 1–31. http://www.educationaldesigner.org/ed/volume4/issue13/article50/

Gravemeijer, K., & Cobb, P. (2006). Educational Design Research Educational Design Research. *Netherlands Institute for Curriculum Development: SLO*, 11. http://www.eric.ed.gov/ERICWebPortal/recordDetail?accno=EJ815766

Gravemeijer, K., & Stephan, M. (2002). Emergent Models as an Instructional Design Heuristic. In: Gravemeijer, K., Lehrer, R., Van Oers, B., Verschaffel, L. (eds). *Symbolizing, Modeling and Tool*

Use in Mathematics Education. Mathematics Education Library, vol 30. Springer, Dordrecht. https://doi.org/10.1007/978-94-017-3194-2_10

Hasanah, U. (2023). Innovation of Mathematics Learning Models and Media in Elementary Schools in Kurikulum Merdeka Belajar. *KnE Social Sciences*, 255–265. https://doi.org/10.18502/kss.v8i8.13302

Hollebrands, K. F. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *The Journal of Mathematical Behavior*, *22*(1), 55-72. https://doi.org/10.1016/S0732-3123(03)00004-X

Isnanto, R. R., Hidayatno, A., & Zahra, A. A. (2020, June). Fractal batik motifs generation using variations of parameters in julia set function. In *2020 8th International Conference on Information and Communication Technology (ICoICT)* (pp. 1-6). IEEE. https://doi.org/10.1109/ICoICT49345.2020.9166282

Martin, G. E. (2012). *Transformation geometry: An introduction to symmetry*. Springer Science & Bussiness Media. https://books.google.to/books?id=KW4EwONsQJgC

Mashingaidze, S. (2012). The teaching of geometric (isometric) transformations at secondary school level: What approach to use and why? *Asian Social Science*, *8*(15), 197–210. https://doi.org/10.5539/ass.v8n15p197

Mbusi, N. (2015). *Misconceptions and Related Errors Displayed By Pre-Service Foundation Phase Teachers in Transformation Geometry*. *Luneta*, 386–400. https://uir.unisa.ac.za/bitstream/handle/10500/22894/Nokwanda%20Mbusi.pdf

NCTM. (2020). NCTM Standards (2020) – Secondary (Initial Preparation). *NCTM Standards - Positions*, 1–6. https://www.nctm.org/standards/

Ngirishi, H., & Bansilal, S. (2019). An exploration of high school learners' understanding of geometric concepts. *Problems of Education in the 21st Century*, *77*(1), 82–96. https://doi.org/10.33225/PEC/19.77.82

Noerhasmalina., & Khasanah, B. A. (2023). The geometric contents and the values of local batik in Indonesia. *Jurnal Elemen*, *9*(1), 211-226. https://doi.org/10.29408/jel.v9i1.6919

Plomp, T. (2013). Introduction to Educational Design Research: An Introduction. *An Introduction to Educational Design Research - Part A*, 11–50. http://www.eric.ed.gov/ERICWebPortal/recordDetail?accno=EJ815766%0Ahttp://international.slo.nl/edr

Prahmana, R. C. I., & D'Ambrosio, U. (2020). Learning geometry and values from patterns: Ethnomathematics on the batik patterns of yogyakarta, indonesia. *Journal on Mathematics Education*, *11*(3), 439–456. https://doi.org/10.22342/jme.11.3.12949.439-456

Pratikno, Y., Hermawan, E., & Arifin, A. L. (2022). Human Resource 'Kurikulum Merdeka'from Design to Implementation in the School: What Worked and What not in Indonesian Education. *Jurnal Iqra': Kajian Ilmu Pendidikan*, *7*(1), 326–343. https://doi.org/10.25217/ji.v7i1.1708

Putri, R. I. I., & Dolk, M., Zulkardi, Z. (2015). Professional Development of Pmri Teachers for. *IndoMS-JME : Journal on Mathematics Education*, *6*(1), 11–19. https://ejournal.unsri.ac.id/index.php/jme/article/view/1900/795

Rellensmann, J., Schukajlow, S., & Leopold, C. (2020). Measuring and investigating strategic knowledge about drawing to solve geometry modelling problems. *ZDM*, *52*, 97-110. https://doi.org/10.1007/s11858-019-01085-1

Simon, M. A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education*, *26*(2), 114–145. https://doi.org/10.5951/jresematheduc.26.2.0114

Simon, M. A., Placa, N., Avitzur, A., & Kara, M. (2018). Promoting a concept of fraction-as-measure: A study of the Learning Through Activity research program. *Journal of Mathematical Behavior*, *52*(March), 122–133. https://doi.org/10.1016/j.jmathb.2018.03.004

Sinclair, N., & Bruce, C. D. (2015). New opportunities in geometry education at the primary school. *ZDM Mathematics Education*, *47*(3), 319–329. https://doi.org/10.1007/s11858-015-0693-4

Toole, T. O. (2001). *Building Powerful Understanding by Connecting Informal and Formal Knowledge*. 384–391. https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=5ccf06ee5526de9e90e72ec2fce4466399823f0b

Treffers, A. (1987). Integrated column arithmetic according to progressive schematisation. *Educational studies in Mathematics*, *18*(2), 125-145. https://doi.org/10.1007/BF00314723

Trimurtini., Waluya, S. B., Sukestiyarno, Y. L., & Kharisudin, I. (2021). A Systematic Review on Geometric Thinking: A Review Research Between 2017-2021. *European Journal of Educational Research*, *11*(1), 69–81. https://doi.org/10.12973/eu-jer.11.3.1535

Usiskin, Z. (1982). Van Hiele Levels and Achievement in Secondary School Geometry. In *CDASSG Project*. https://ucsmp.uchicago.edu/resources/van_hiele_levels.pdf

Van den Heuvel-panhuizen, M., Drijvers, P., Education, M., Sciences, B., & Goffree, F. (2020). Encyclopedia of Mathematics Education. *Encyclopedia of Mathematics Education*. https://doi.org/10.1007/978-94-007-4978-8

Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Academic press. https://cir.nii.ac.jp/crid/1130000795426623744

Vojkuvkova, I. (2012). The van Hiele model of geometric thinking. *WDS'12 Proceedings of Contributed Papers*, 1, 72-75. https://physics.mff.cuni.cz/wds/proc/pdf12/WDS12_112_m8_Vojkuvkova.pdf

Walukow, M. R., Naharia, O., Wullur, M. N., Sumual, S. D. M., & Monoarfa, H. (2023). Implementation of Merdeka Belajar Policy: Constraints in the Pancasila Students Profile Strengthening Project. *International Journal of Multidisciplinary Approach Research and Science*, *1*(02), 104–116. https://doi.org/10.59653/ijmars.v1i02.62

Zidan, M. R., & Qamariah, Z. (2023). A Literature study on the implementation of merdeka curriculum. *Jurnal Riset Rumpun Ilmu Bahasa*, *2*(2), 153–167. https://doi.org/10.55606/jurribah.v2i2.1576

Zulkardi. (2002). *Developing A Learning Environment On Realistic Mathematics Education For Indonesian Student Teacher (Doctoral disertation, University of Twente, Enschede)*. 1–218. http://doc.utwente.nl/58718/1/thesis_Zulkardi.pdf