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Linking diversity in learning Geometry: Exploring tessellation in techno-based mathematical tasks

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Abstract

Nowadays, digital technologies are crucial in supporting students in geometry in secondary mathematics classrooms. However, in some cases, the role of visual function in technology was only utilized for seeing and conjecturing, not for experimenting, while to develop a relational understanding of geometry concepts, students should actively participate in the learning process. To address the issue, this study investigated how students learn geometry with digital technology assistance based on students' diversity in their mathematics abilities. A task with a dynamic geometry software called Techno-based Mathematical Tasks (TbMT) was designed to assist students in exploring geometrical activities and solving a problem through investigations on tessellation. This research employs educational design research and focuses on the preliminary design by conducting a pilot study on three students based on the diversity in their ability in mathematics classrooms, i.e., low, middle, and high. As part of data collection, we captured students' works to examine critical information in their responses based on their differences in abilities. We collected the data through online meetings and recorded the data. We analyzed students' work from the recording by capturing critical information. The results revealed that the TbMT might provide students with opportunities to learn by exploring tessellation activities that might contribute to students' understanding of geometry concepts. Due to the limited number of participants in this study, further research can be an opportunity to expand the number of participants to enhance the contribution to the literature with more comprehensive empirical evidence.

Keywords: Design Research, Dynamics Geometry Software, Geometry, Students' Diversity, Tessellation

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Geometry is a strand of mathematics that is used in various fields in real-world contexts. However, some studies reported that students still face challenges in dealing with geometry concepts. As Ngirishi and Bansilal (2019) pointed out in their research, they found that most students who participated had difficulties solving a geometric reasoning problem. Students' mathematical reasoning, spatial, and argumentation skills were low when learning geometry, especially when working with problem-solving questions (Cesaria & Herman, 2019; Hendroanto et al., 2018; Sukirwan et al., 2018). Furthermore, Ng et al. (2020) and Ozdemir (2010) discovered that the main challenging problems in geometry topics for students are visualization and mental transformation related to develop students' spatial skills. Therefore, there is a need to facilitate students in learning geometry efficiently.

Besides the challenges in learning geometry, many studies reported that integrating digital technologies plays a vital role in supporting learning and enabling students' practical exploration of geometry topics. In a study conducted by Kurtulus and Uygan (2010), they also revealed that digital technologies-based activities positively impact students' skills, especially spatial skills. Another study conducted by Fonna and Mursalin (2019) also demonstrated the effect of geometry software on promoting students' mathematical representation. Furthermore, some authors also agreed that the interactive activities by utilizing dynamics geometry software might elevate students' learning (Ghavifekr & Rosdy, 2015; Ra et al., 2019; Temsah & Moukarzel, 2018). Thus, in this study point, it is crucial to consider using geometry software to facilitate students' learning.

The novelty of this study is structured into three different facets. First, as previous research conducted by Cerrone (2006) and Bays (2005) which utilized the tessellation of pentagons to provide learning activities for students of various ages. However, to the best of our knowledge, there is no information on the use of digital technologies to support student's learning in this context. Second, there are opportunities to utilize tessellation in supporting students learning geometry (Tikoo, 1998; Gökdağ et al., 2022). Third, as previously discussed, there were found challenges and difficulties for students in learning geometry especially in visualization and reasoning (Hendroanto et al., 2018; Ngirishi & Bansilal, 2019; Cesaria & Herman, 2019; Ng et al., 2020; Ozdemir, 2010). Therefore, our research focus on facilitating students learning of geometry through digital technologies by utilizing dynamics geometry software with tessellation.

By tailoring techno-based mathematics tasks with dynamics geometry software to the students' diversity, we aimed to support students in managing their cognitive load in learning geometry. This study used educational design research to explore students' responses to the developed task. We aimed to contribute to the educational field by providing valuable insights on providing Techno-based Mathematics Tasks (TbMT) that can help to address students' learning diversity. More specifically, we explored how different tasks or combinations can contribute to learning. To this end, the following research question was formulated: *What differences are found in students' exploration of techno-based mathematics tasks based on students' diversity of mathematical ability in learning?*

The Implementation of Geometry and Tessellation in Learning Activities

An author revealed that experimental activities should be integrated into learning processes (Laborde, 2001). Moreover, Leung (2011) underlined the importance of several features in dynamics geometry software that can empower students' skills to explore geometric objects, facilitating students to experiment. For instance, Ng et al. (2020) argued that dynamics geometry software could enhance visualization works more interactive by dragging, rotating, and changing views which is essential to enable students' exploration and experimentation.

As one of the mathematics strands, geometry is used in various fields closely related to real-world contexts and has played significant roles since long ago (Silmi Juman et al., 2022). Silmi Juman et al. (2022) pointed out that one of evidence that can be seen today was created by Ancient Egyptians, who used geometrical concepts to construct buildings with remarkable shapes and structures. Furthermore, a study conducted by Tatsuoka et al. (2004) reported a significant relationship between geometry activities and any attributes characterizing high order thinking skills in mathematics learning. Their study also showed that geometry enhances junior high school students' mathematical thinking skills. However, as reported by several researchers, most students face difficulties in learning geometry because geometry requires high-level thinking skills, which insist students gradually enhance their skills (Cesaria & Herman, 2019; Hendroanto et al., 2018; Sukirwan et al., 2018; Prahmana & D'Ambrosio, 2020)

In this study, we integrate tessellation as a topic to facilitate students learning geometry with dynamics geometry software. Our study corroborated by two studies which presented the opportunities of tessellation in mathematics activities (Tikoo, 1998; Gökdağ et al., 2022). Incorporating tessellation as an exploration in learning geometry for students which can be a good topic to discuss congruency and defined that tessellation consists of polygons covering an area without overlapping or leaving any gaps that create a pattern (Tikoo, 1998). Gökdağ et al. (2022) reported that students who experienced tessellation activities acknowledged the aesthetic value and exhibited persistence and self-motivation to finish the activities. Additionally, tessellation context might provide several benefits. As underlined by de Villiers (1996), the context offers intuitive visualization which is in line with geometry content in aesthetic patterns, namely angles, lines, and other geometric properties. Moreover, to develop the TbMT, it was referred to the discussion on the pentagonal tessellation presented by Bays (2005).

A study conducted by Tatar et al. (2014) revealed the positive effect of using dynamics software in learning geometry. Another study also indicates the positive contribution of dynamics geometry software in mathematics classrooms which is useful to create and transform any geometrical objects (Falcón, 2011). Furthermore, we took this as an opportunity to utilize dynamics geometry software in tessellation activities, especially to facilitate students with easier manipulation and transformations of polygons inspired by Falcón's research (2011). In our developed activities, we utilize dynamics geometry software to facilitate students with examples and non-examples of tessellation by using pentagons. The idea was supported by Avcu (2022), who reported that the importance of non-examples, but sadly teachers rarely used them as activities.

Students' Diversity

Students' diversity may vary and influence students learning, which challenges teachers to promote students' opportunities to learn that can optimize the potential of every student. As suggested by (Goedhart et al., 2019), educators have been challenged to respond with the innovation of their teaching methods in their daily classrooms. Additionally, student diversity is essential to be considered by teachers to meet their students' needs (Felder & Brent, 2005). Moreover, as revealed by Prediger and Buró (2021) in their study, addressing students' diversity should be considered by teachers by providing rich resources. For instance, Abeysekera and Dawson (2015) argued that teachers might integrate activities that facilitate students learning at their own pace and outside the classroom settings.

In the recent decade, some studies reported the significance of conducting learning, considering students' cognitive diversity based on their learning abilities. This argument supported by Groher et al. (2022) suggested that students' diversity is more than gender, it might be related to students' educational background and prior knowledge. An investigation revealed the importance of promoting mathematics learning that contemplates future action on students' diversity to students' autonomous learning and mathematical performance (White, 2003). White (2003) suggested enhancing mathematics learning considering students' diversity of mathematical performance. She also underlined that the main objectives of the lessons developed within the project were to encourage students' innovative and ingenious problem-solving activities. Moreover, a study demonstrated the effect of diversity of students' ability (high- and low-ability) in giving feedback, providing criticism, and focusing on problem-solving in writing classrooms (Patchan et al., 2013). Furthermore, two studies conducted by van Garderen et al. (2013) examined how students with diverse abilities in problem-solving performed in solving mathematics word problems. Therefore, we considered the students' diversity in our investigations based on significant observations that affected students from the previous studies. Based on the aforementioned, in this study, we focused on contributing to students in learning geometry by focusing on students' diversity. What we meant by diversity in this study context is students' mathematics ability which is presented in the following section.

Techno-based Mathematics Tasks

In this study, we developed tasks in geometry topics with tessellations context using dynamics geometry software. In developing the task, we wanted to facilitate students' thinking by managing their cognitive load. Clements and Battista (1992) underlined the importance discontinue processes in learning geometry as the characteristics of geometric thinking. Fuys et al. (1988) underlined the importance of students' thinking levels in learning geometry. Usiskin (1982) underlined that based on Van Hiele's theory, there is a need to go through certain levels in understanding geometry. Fuys et al. (1988) and Usiskin (1982) mentioned that the levels of geometry thinking based on Van Hiele's theory start with recognition (Level 0), analysis (Level 1), order (Level 2), deduction (Level 3), and rigor (Level 4). Therefore, the developed tasks sought an understanding of the importance of students' level of geometric thinking skills. For instance, in the first task (preliminary task), the TbMT focuses on the student's recognition of the problem, continuing by analysis and order for the second activities (challenge tasks). The TbMT facilitates students to analyze figures and evaluate the relationships by exploring second activities. Then, following the Van Hiele's theory, the TbMT provides students with opportunities to prove and establish interrelationships to build theorems on which the pentagon can be tessellated and what is the reason behind it by exploring the third task (closure activity). We provide a description of our development activities and the design task in the following sections.

METHODS

The following sections describe the methodology, research aim, data collection and processing in more detail.

Research Aims

This study aims to support students in managing their cognitive load in learning geometry by tailoring Techno-based Mathematics Tasks (TbMT) with dynamics geometry software (DGS) to the students' diversity. This study's purpose to investigate students' different responses in order to explore students' responses to the self-developed task. Furthermore, aiming to contribute to the educational field by providing valuable insights on providing TbMT, which may help address students' learning diversity. More specifically, it explored how different tasks or combinations can contribute to students' learning. Therefore, the following research question was formulated: What differences are found in students' exploration of techno-based mathematics tasks based on students' diversity of mathematical ability in learning?

Research Design

This investigation employed an educational design research framework which refers to McKenney and Reeves (2020) and Bakker (2018). The following design research activities consisted of three main stages, as shown in Figure 1. In the initial stage of our project, the authors tried to conduct a literature review and analyze the Indonesian national curriculum standard to align with the topic and context for

Indonesian students. Based on the analysis, the TbMT was appropriate for eighth-grader students who already have learnt triangles and quadrilaterals. It was found that tessellation has favorable prospects for implementation in geometry topics by utilizing DGS in the Indonesian curriculum. Accordingly, after the authors made sure of the conformity of the curriculum, then TbMT was developed to foster students' learning of geometry through tessellation.

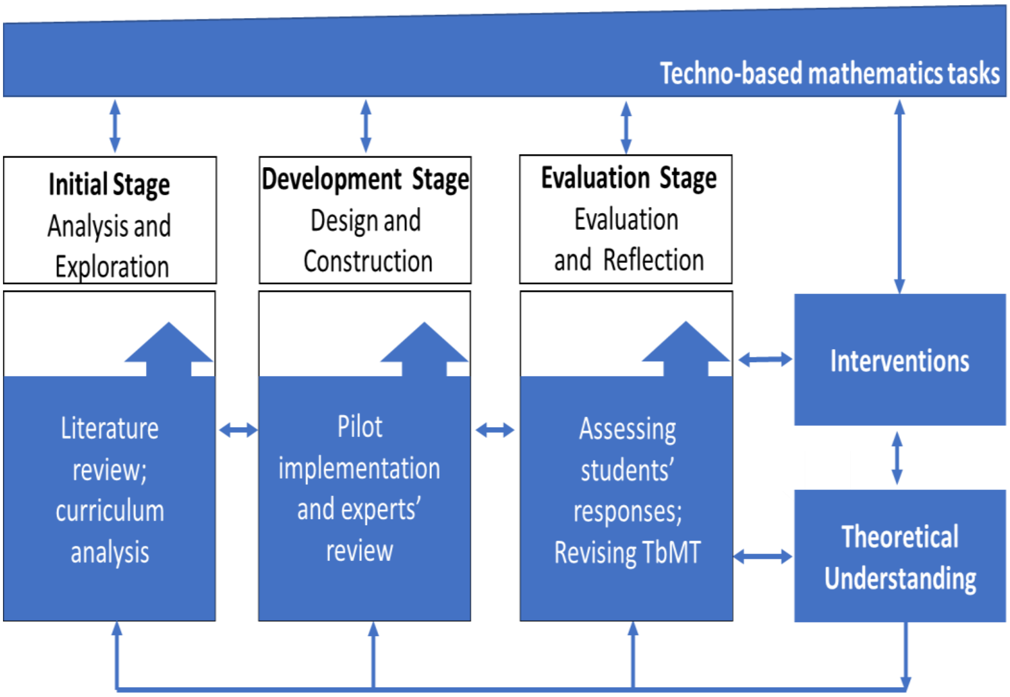


Figure 1. The TbMT educational design research processes followed in this study (adapted from McKenney & Reeves, 2020)

This study was the result of the development stage in which in the initial development stage the authors designed the prototype of TbMT. Therefore, in this article we only describe and focus on the developed tasks and the initial stage of design research. The authors developed TbMT using one of the widely used DGS called Cabri 3.3 as suggested by many researchers because of its contribution to facilitating students' learning in mathematics classrooms, especially geometry (Klasa, 2010; Laksmiwati, 2015; Straesser, 2001; Tutak et al., 2009). The TbMT was designed as task sequences that students can explore for a duration of 45 minutes. The task sequences comprise eight activities: a preliminary task, a challenge task, five main tasks and a closure exercise. The activities were designed to enhance students' exploration, manipulation, observation, and construction. Afterwards, the authors asked three experts from the mathematics education field to review and give their constructive feedback. Some revisions have been executed based on reviewers' suggestions (e.g., giving clearer instructions for children, revising some sentences, and revising the size of the activities that the original version was too small).

Sampling

Ultimately, the authors conducted a pilot implementation which involved three eighth-grader students from a public school in West Java, Indonesia collaborating with a mathematics teacher. This pilot implementation is a small-scale implementation which was a significant stage of our research activities within the design research stage utilized in this study. Therefore, this study will be utilized to support the next stage of larger-scale implementation. The students have been selected by the teacher based on their performance in learning mathematics which is divided into three levels (high, middle, and low performance). Based on her observations and assessment, the teacher was asked to evaluate students' performance in mathematics classrooms. Other researchers have utilized this method that relies on teachers' observations in choosing study groups (Moreau & Coquin-Viennot, 2003; Voyer, 2011). The teacher selected students from three groups: high-, middle-, and low-ability of mathematics performance in learning. Thus, in this study, we utilize purposeful sampling methods, which rely on the teacher's decision regarding her students' evaluation. Purposeful sampling was suggested by Şahin et al. (2016) and Patton (1987) to get rich information about a case based on the situation and condition of the research objects. We need a certain condition based on students' level of mathematics performance to capture a deeper understanding of students' diversity and their responses.

Moreover, the implementation was conducted remotely, so students joined an online conference meeting from their homes. We controlled students' discussions by working collaboratively with their teachers and parents to make sure that students actively engage in the activities. For example, asking their parent to observe without giving any intervention and asked them to provide documentation The authors documented their responses and conducted an analysis with comparing their responses based on students' mathematical performance as we focused on students' diversity. Figure 2 demonstrates the home activities.

Figure 2. Students' activities while working with the developed tasks

Data Collection and Analysis

In this study we collected the data through conducting online meetings with teachers and the participants of this study. One of the researchers lead the session and in the main activities, students were given opportunities to explore the tasks. They have been informed that the lesson will be recorded. Then, we analyze students' work based on their diversity to capture important information from students' different responses based on their mathematics ability. We present the results and discuss them to connect with existing literature in the following section.

RESULTS AND DISCUSSION

Turning to the activities that have been developed, the authors expected students to conduct individual exploration. The main objective of the activities is to promote students' experimental activities through the TbMT consisting of eight activities. Three students successfully participated in our study. They were asked to solve the tasks in a tessellation context aligned with the Indonesian curriculum in Geometry topic. The TbMT provides several geometric features for supporting students problem-solving activities. For instance, while exploring the activities, students might do drag and drop, rotate, write notes, create segments and angles, measure segments and angles, etc. The researchers intentionally provide the features to support students' investigations. In the following, this study's qualitative results and discussion are described in detail.

Task Design

One of the researchers described some technical matters to the students in the implementation. The researcher introduced TbMT and how to explore it with Cabri 3.3, mainly discussing the tessellation context that should be done in the tasks without excessive interference with the main activities. The initial actions are the preliminary and challenge tasks, as shown in Figure 3. The first activity was designed to intrigue students and introduce tessellation. Moreover, the preliminary task is crucial to introduce some features of the TbMT, followed by the challenge task. The challenge task asked students to use tessellation with triangles, squares, and hexagons. Therefore, the tasks aimed to stimulate students to tessellate two or more polygons as stepstones to the main task.

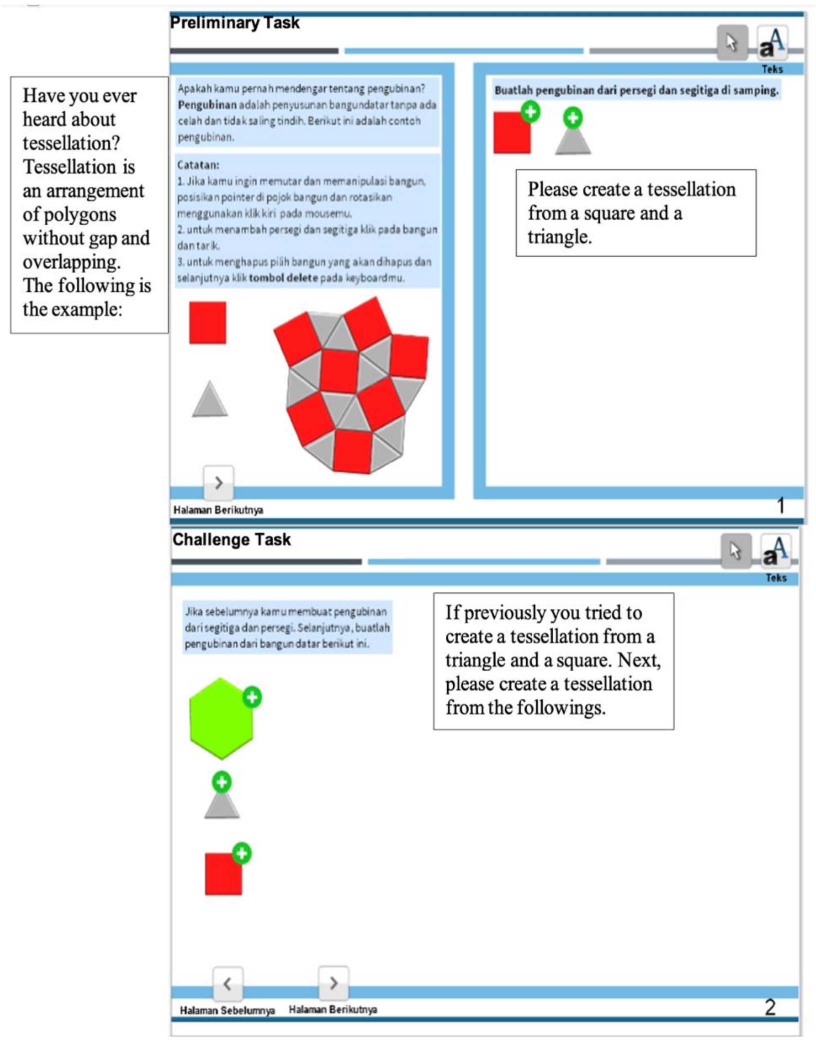


Figure 3. The initial tasks consist of the preliminary task and the challenge task

The subsequent activities are five main tasks that start with prompting questions related to pentagons. In these activities, four pentagons are presented. Students are asked to create a conjecture of which pentagon can be tessellated before exploring. Afterwards, they are asked to explore by themselves to prove their conjecture with the following activities. In these explorations, students are not allowed to revise their conjecture (the first main task). The following activities aim to facilitate students' experimentation, so they can find which pentagon can be tessellated. The next four tasks can be tackled non-sequentially, allowing students more flexibility in conducting backtracking. The main tasks are illustrated in Figure 4.

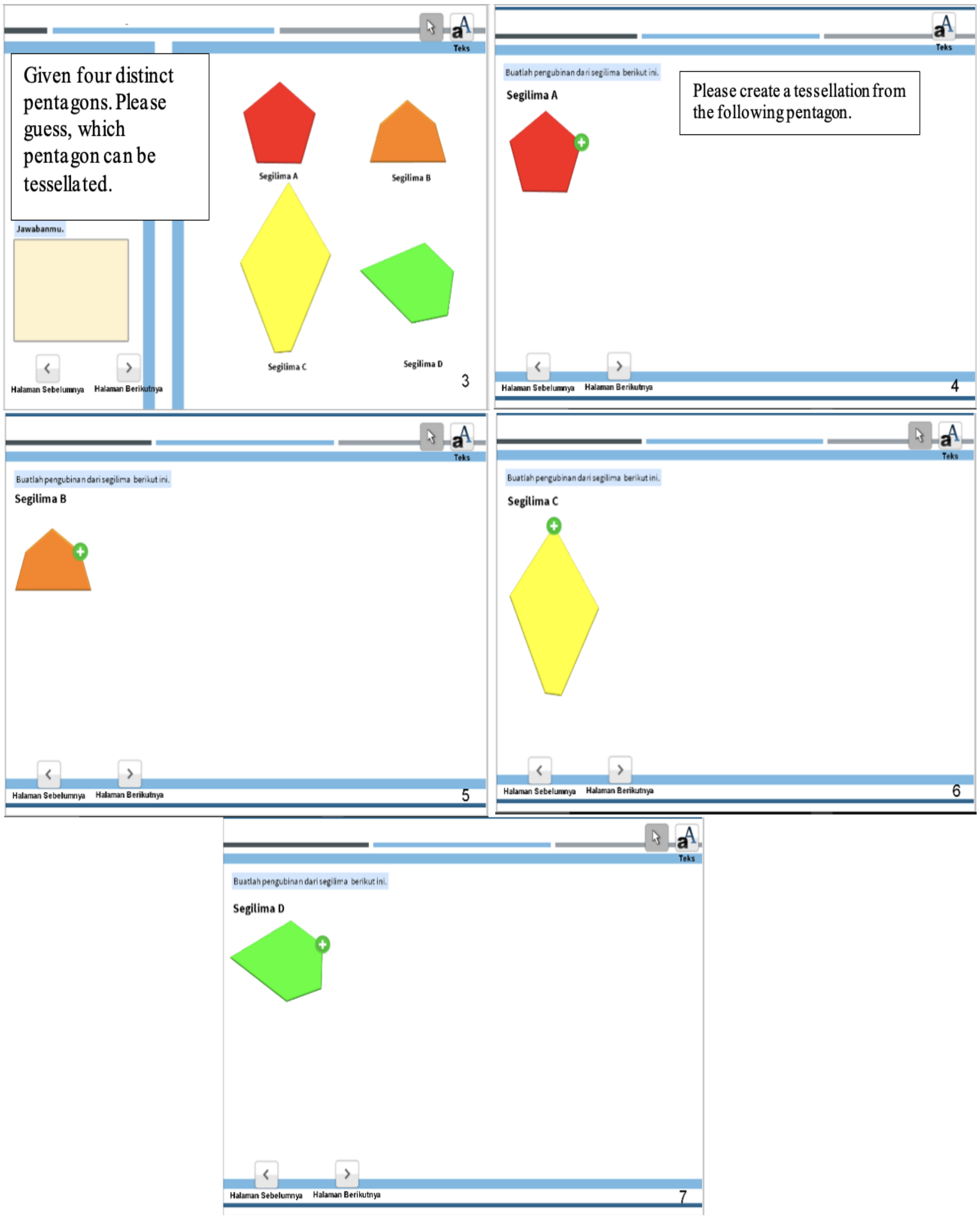


Figure 4. The main tasks consisted of one prompting question and four experiments

The last activity is the closure task which focuses on generating a conclusion based on the previous main tasks (as can be indicated in Figure 5). There are two main questions in this task which evaluate students' understanding to showcase their experiment activities. The primary purpose of the first questions in this activity is to report their findings. Moreover, the second question asks students to explain what factor might potentially influence and determine whether a pentagon can be tessellated or not. Therefore, the closure task plays a vital role in the developed activities in observing and indicating students' understanding comprehensively.

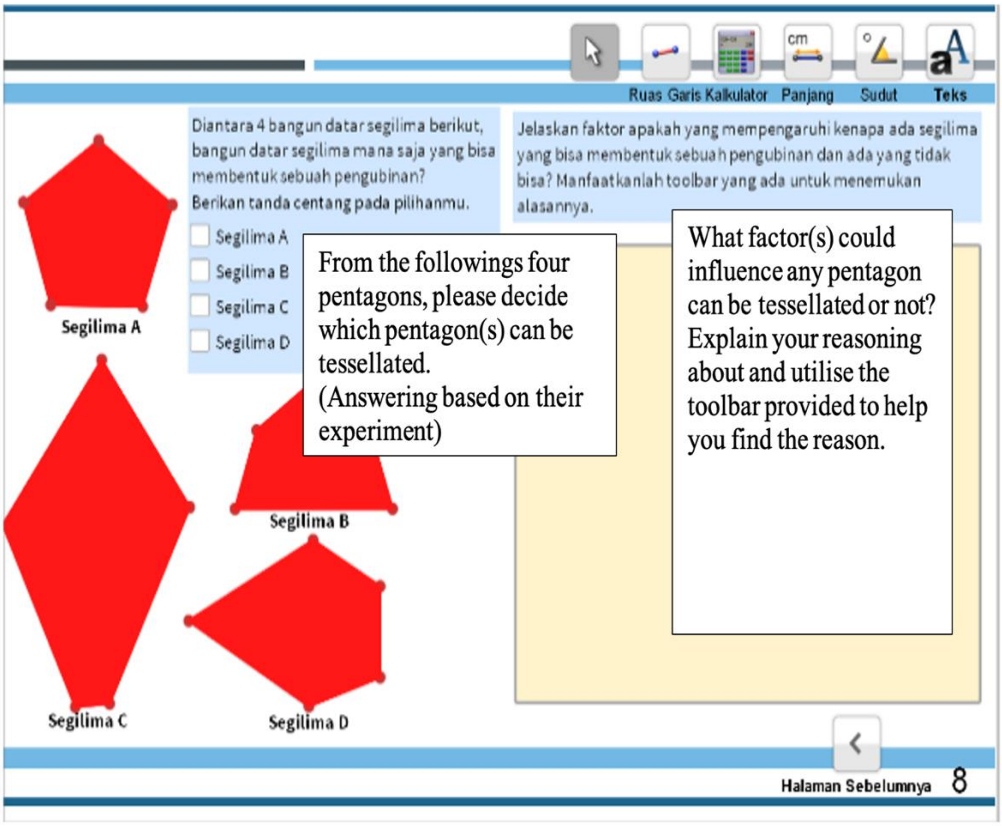


Figure 5. The closure activity

Students' Responses based on Their Diversity of Mathematics Performance

The main issue of this study was to investigate the differences which could be found in students' exploration of techno-based mathematics tasks based on the diversity in students' mathematics performance. This section discusses our findings and compares students' work based on their mathematics performance. Therefore, the authors present students' work on the preliminary task, the challenge task, the main task, and closing with the closure task. Hence, the authors might comprehensively understand students' responses based on their diversity to answer the research questions.

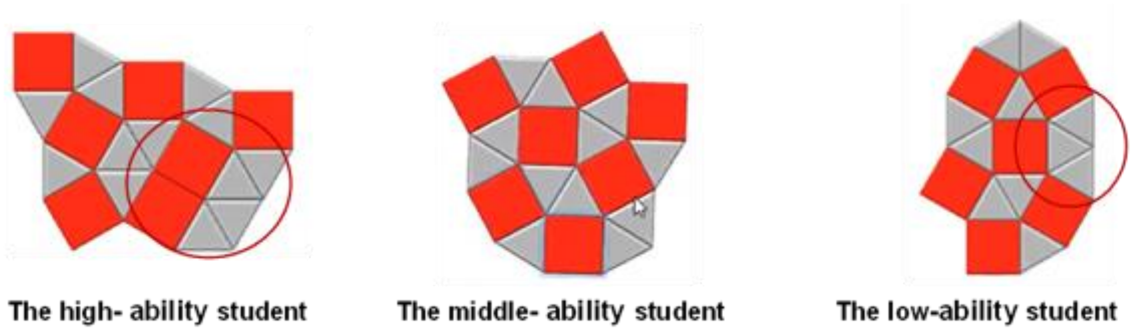


Figure 6. Students' responses to the preliminary task

Based on Figure 6, it can be indicated that not all students followed the instructions. Astoundingly, it can be found that only the student with middle-ability successfully tessellated the two regular shapes provided, which are like the provided pattern. However, the high and low-ability students seemed to be trying to create different ways. Unfortunately, the tessellation created by the high and low-ability students cannot be infinitely repeated as it might produce overlaps and gaps, as shown in Figure 6 (see the red circled area).

Moreover, the students were presented with a challenge task, which required them to create the tessellation using three regular shapes, that is, square and hexagon. This activity allowed students to experience and explore more about tessellation. As in the previous task, they were asked to produce their pattern without guidance. During this activity, most of the students were doing trial and error activities before finally finishing their tessellation. Figure 7 presents students' work on the challenge task.

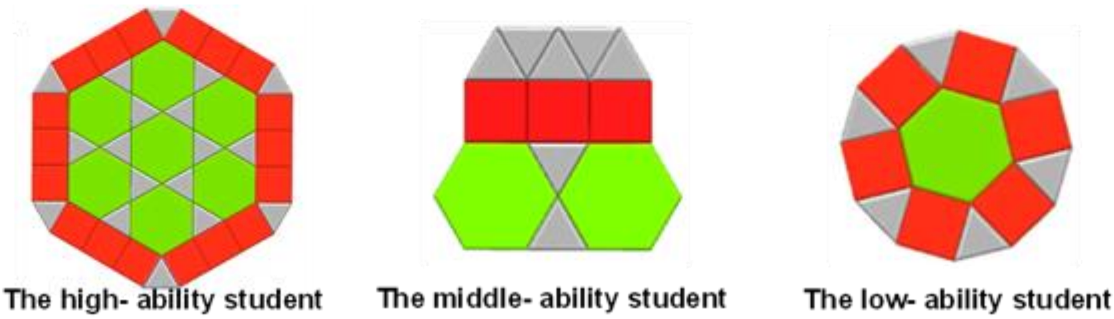


Figure 7. The example of students' answers to the challenge task

Figure 7 shows the pattern created by the high- and low-ability student does not belong to repeated pattern tessellation because some overlaps or gaps appear if the work is being continued. All students had not learned about tessellation previously, so, understandably, they struggled at the beginning of the lesson. Additionally, based on the pilot implementation, students did not have significant difficulties exploring the TbMT, even if it was their first experience dealing with Cabri 3.3. Even though the high- and low-ability students made a mistake in the first task. Based on our observation, the students also understand the meaning of tessellation and its property. For instance, one or more shapes possess tessellation attributes if they can be formed as a repeated pattern without overlaps or gaps. Therefore, this study provides valuable insights that can contribute to further explorations.

Based on the students' works in Figure 7, all students could create their own patterns. However, since they only made the pre-images that did not fill the entire screen, the students could not see whether the designs could be continued to cover the whole plane and could be repeated in every direction. Some of the patterns created did not belong to semi-regular tessellation because the arrangement of the polygons at each vertex is not the same. Notably, to create a semi-regular tessellation formed by two or more regular polygon, the arrangement of the regular polygons around any vertex must be the same (Chakraborty & Caglayan, 2017; Watson, 1973). It can be indicated that high-ability students tried to create complex patterns. However, the middle-ability students created the less complicated but still unsuccessful tessellation. Figure 7 shows the low-ability student who successfully completed the semi-regular tessellation. Thus, the findings and discussion are becoming intriguing to note that the low-ability student only succeeded in dealing with the second problem.

Moreover, to answer the research questions, in this section, the authors discuss how this study revealed the differences among students' diversity based on mathematics performance in learning mathematics. The findings pointed out that there are different responses demonstrated in Figure 6 and Figure 7. As shown in the figures, students with various mathematical performances respond with varying levels of complexity. This finding aligns with Voyer's (2011) statement that students with weak, average, and strong mathematical skills get different mean scores in solving mathematics problems. However, interestingly, based on the figures, the authors found that students with low level of performance who the only students that could solve the second problem. The middle-ability student incorrectly answered the challenge task (see Figure 7), which indicated the student made a mistake on one of the questions. While the student with a high level of performance made mistakes on both questions, he inaccurately solved the first two questions. This finding is supported by a study conducted by Hariati and Septiadi (2019), who reported that high-ability students experienced more mistakes than lower-ability students. Students with lower ability tends to have errors in transformation problem, processes, and writing final answer (Hariati & Septiadi, 2009). In contrast, the result is incongruous with the finding obtained by Voyer (2011), who noted that high-ability students are likelier to make fewer mistakes. Hence, for further explorations, it is essential to clarify and investigate how students' different levels of mathematics performance influenced their work.

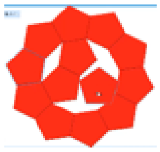
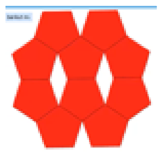
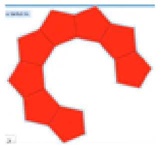
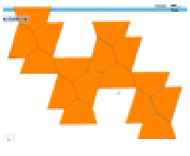
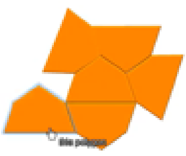
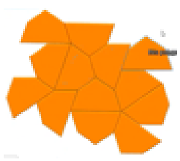
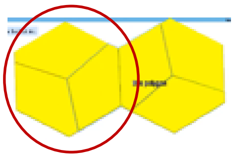
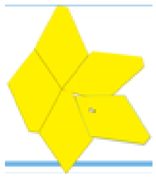
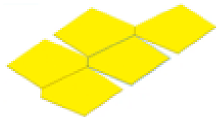
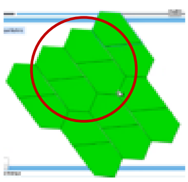
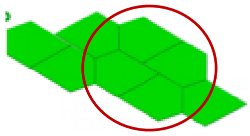
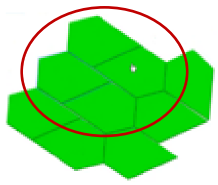
In the subsequent activities, the students engaged with the main tasks that consisted of four sub-activities. Before exploring the sub-activities, the students were asked to predict which pentagon(s) could be tessellated from four given pentagons (one regular pentagon and three irregular pentagons). Regardless of their mathematical ability, most students predicted that the pentagon A (a regular pentagon) could be tessellated. The authors presented students' responses in the Figure 8.

Figure 8. Students' predictions on a pentagon(s) that can be tessellated

After writing the prediction, the students continued to explore and investigate the four pentagons. Table 1 reports the students' responses to the four main tasks. From Table 1, it can be indicated that all students agreed that Pentagon A and Pentagon B could not be tessellated. However, Pentagon C has been found by the high-ability students that can be tessellated. Noteworthy, all students have found that it is possible to tesselate pentagon D.

Cerrone (2006) noticed that any polygon formed by combining a triangle and a rectangle with congruent bases would produce a tessellating pentagon. She presented an example that Pentagon C and Pentagon D were two pentagons that can be tessellated. Remarkably, the high-ability students who could not give correct answers for the preliminary and challenge tasks were the only students who answered correctly. He found that Pentagon C and Pentagon D are two pentagons that can be tessellated, as reported in Table 1.

Table 1. The main tasks and students' responses

Students' ability of mathematical performance			Notes
High	**Medium**	**Low**	
			Pentagon A: All students found Pentagon A cannot be tessellated. There are gaps or overlapping pentagons.
			Pentagon B: All students found Pentagon B cannot be tessellated.
			Pentagon C: Only the high-ability student found that Pentagon C can be tessellated.
			Pentagon D: All students found that Pentagon D can be tessellated, but they created three different patterns.

Referring to Table 1, the high-ability student correctly answered Pentagon C and Pentagon D as two pentagons which display the attribute of being able to be tessellated. Based on the observation, he looked back to the previous Pentagon A and did a similar thing with the two other pentagons (pentagon B and pentagon C). He did similarly to pentagon B but failed in making a tessellation from the pentagon (see Table 1, row 2). Finally, he found that the pentagon C can be tessellated by creating a hexagon by combining three shapes of Pentagon C (see Table 1, row 3). The finding reported that the high-ability students might consider previous activities as necessary information. For instance, he learned from his failures to solve other tasks. The result is in line with a study by Moreau and Coquin-Viennet (2003), who found that students' mathematical abilities influence the selection of information. They underlined that low-ability students seem to select less solving information than the higher-performance students, which can be found from the finding from this study (see Figure 6). This finding in line with Cerrone (2006), who showed examples of tessellating pentagon which form tessellating hexagon as result of merging two or more irregular pentagons. In her study, she noticed that any polygon which is formed by combining a triangle and a rectangle with congruent bases will produce a tessellating pentagon, as illustrated in Figure 9.

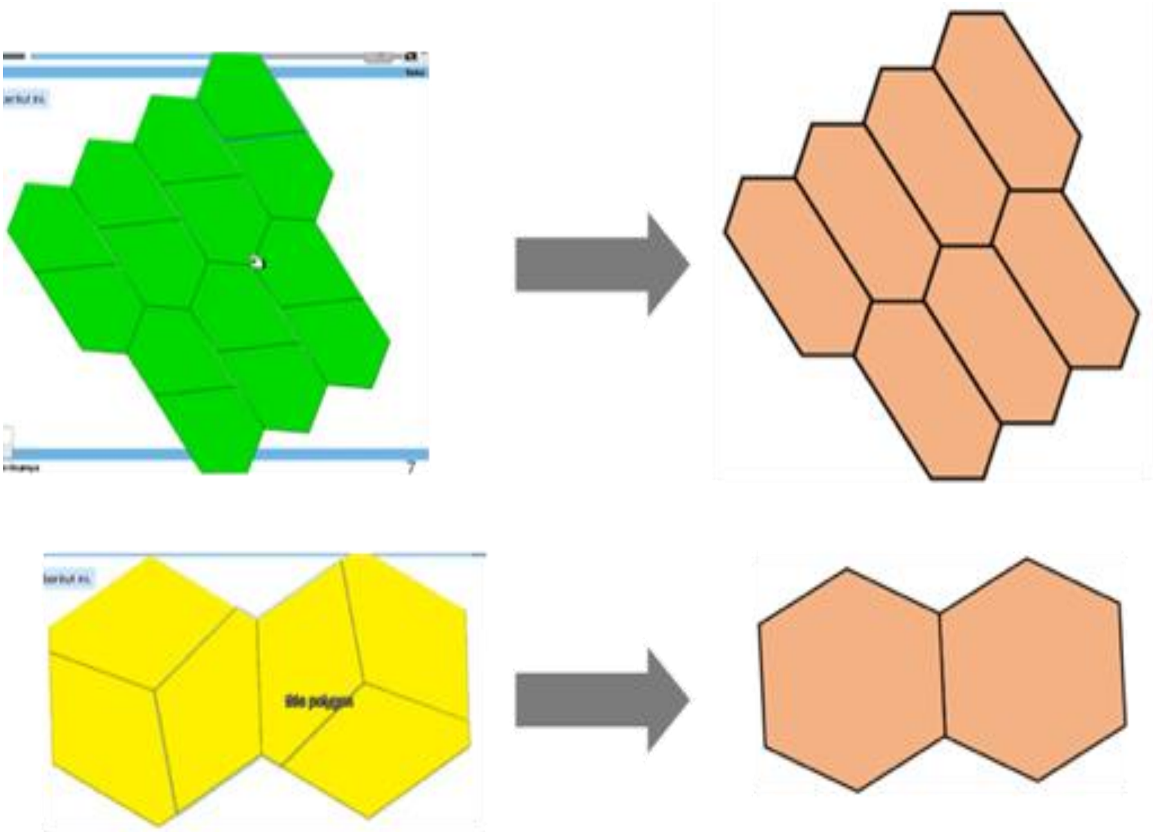


Figure 9. Tessellating pentagon into hexagon (Cerrone, 2006)

Based on the exploration and investigation conducted in the previous phases, the students were encouraged to determine which pentagon(s) could be tessellated and had to explain the reason for their answers. Moreover, this last activity is crucial to stimulate students' reasoning skills illustrated in Figure 10.

Based on Figure 10, there are different responses came from students when we asked what factor(s) that could influence the tessellations of pentagons. We asked students to explain their reasoning about and utilize the toolbar provided to help students to find the reason. The high-ability student answered Pentagon C and Pentagon D, which can be tessellated. He argued, "A pentagon cannot form a tessellation unless transformed into another shape (translation from Figure 10)". Meanwhile, the middle ability student reasoned, "The factor that makes pentagons able to form a tessellation is because they have the same angles, and the factor that makes pentagons unable to form a tessellation is because they do not have the same angles (translation from Figure 10)". This is in line with the low ability student who stated "Angle factor (translation from Figure 10)" as a reason. The responses indicated that the middle-ability and low-ability students notice angles as a factor that could be influenced. Astonishingly, the high-ability student who successfully found the correct answer could not explain the detailed reason for his correct answer.

Concerning the previous study, this study may provide new insight which contributes to the continuation of research conducted by Bays (2005) and Cerrone (2006) with the use of TbMT to support students learning geometry by exploring tessellation in dynamics geometry software. More importantly, in relation to Van Hiele's theory (Fuys et al., 1988; Usiskin, 1982) the TbMT also support students' exploration with go through their students learning. The TbMT provide opportunities to students to experience tasks which focus on geometric thinking levels recognition (Level 0) (see Figures 3 and 6), analysis (Level 1) (see Figures 4 and 7), order (Level 2) (see Figure 4 and Table 1), deduction (Level 3)

(Figure 9), and rigor (Level 4) (Figure 10).

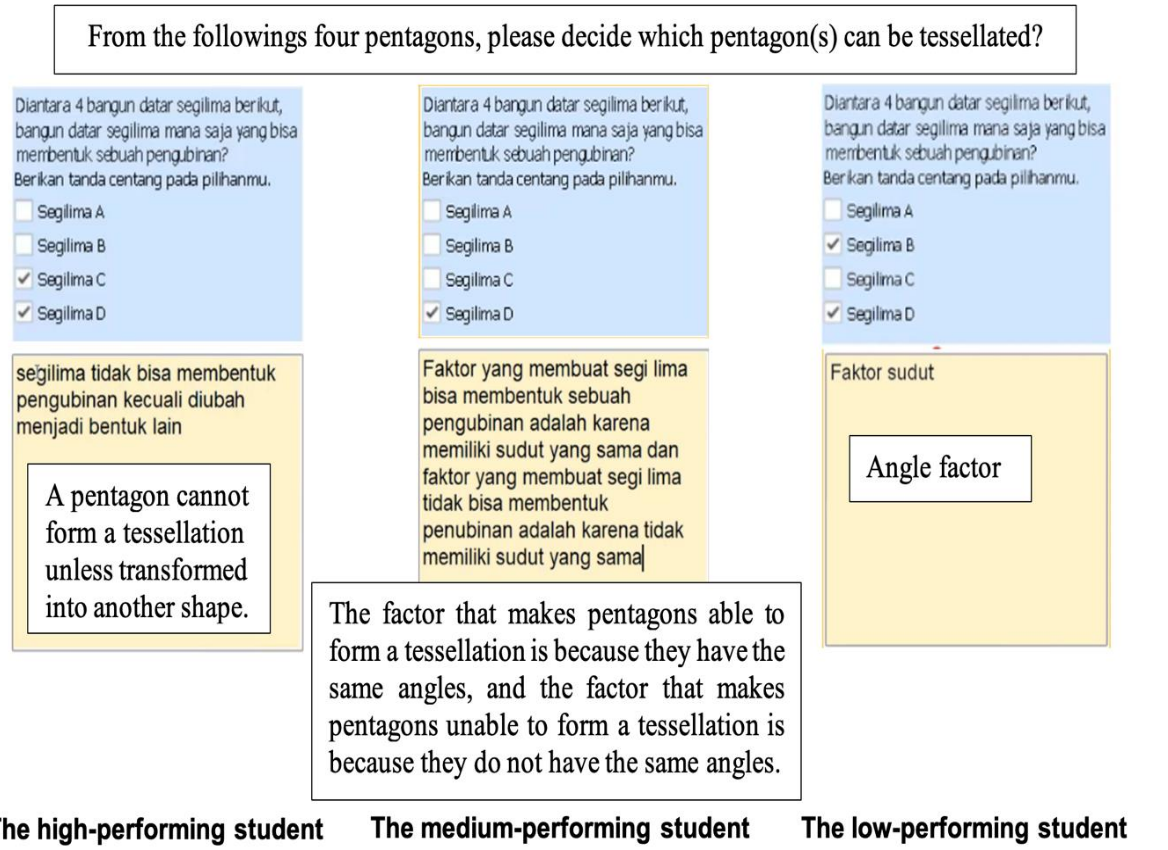


Figure 10. The results of students' answers to the closure task

The findings reported that all students who participated in this study faced challenges in reasoning, which can be denoted by referring to Figure 9. This finding is supported by some studies that reported many students still face challenges in dealing with geometric reasoning, as noted by Ngirishi and Bansilal (2019). Other studies also reported that students' mathematical reasoning (Cesaria & Herman, 2019), spatial (Hendroanto et al., 2018), and argumentation (Sukirwan et al., 2018) skills were found to be problems for students. Furthermore, most of the students in this pilot study focused on the angle as the main reason. While the reasons are not only relying on angles but also the dimension of the pentagons. These results might occur because most students did not use the tools provided at the screen's right top. Therefore, for further implementations, it is crucial to underline the tools which can support their reasoning. This approach aims to assist students in identifying the tools as clues and establishing connections with the prior investigations to solve the problem and generate a conclusion.

Hence, in further study, it is essential to prepare appropriate scaffolding for students, especially for lower-ability students. For instance, teachers might have to make sure that every student will use previous information to generate their conclusion in solving problems. In a study that supports this argument, Hulse et al. (2019) demonstrated in their manuscript that facilitating low-ability students to be more engaged in problem-solving activities gained more benefits. The low-ability students were found to complete more problems. However, Goldman (2009) stated that a lesson must be more adaptive and support students' interactions to promote understanding of students' characteristics in learning. Thus, for further implementation, the tasks might be better implemented in mathematics classrooms, not only as individual explorations.

CONCLUSION

In general, the results from this study's point of view provide promising practices for teachers in supporting students learning with techno-based mathematics tasks (TbMT). It provided choices of learning opportunities for students to explore and construct geometry tasks through tessellation context which focus on supporting students learning and geometric thinking. TbMT aims to enhance students' understanding of geometry concepts and extend their experience by exploring problem-solving activities through open-ended tasks. The tasks gave opportunities for all students, regardless of their mathematical ability, to analyze the tasks based on their understanding. The results of this study are expected to provide a valuable contribution as this study has revealed that technology has supported students with different abilities to learn geometry.

Additionally, the authors found differences between students with low, middle, and high performance based on students' responses in solving TbMT. The high-ability students could do better than the lower-ability students in solving mathematics problems. However, this result is not generalizable as only three students participated. Further research might explore how geometry software can support students learning geometry in more expansive areas with more students. The authors will continue the following design research on the TbMT, reflecting on some findings and assessing the development to continue to the larger scale implementation to gain more comprehensive empirical evidence. For example, consider how to stimulate students' mathematical reasoning skills by giving adequate scaffolding for students, and it is also essential to prepare appropriate scaffolding for students, especially for lower-ability students.

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Declarations

Author Contribution	:	PAL: Developing the techno-based mathematics tasks, designing the study, analyzing the data, and writing the manuscript. MH: Brainstorming the tessellation ideas, collecting, and analyzing the data and writing the first report with some literature reviews after implementation. ES, RCIP, BS, and ZL: Substantial contributions in writing the manuscript and revising the manuscripts. All authors agree with the final version of the content and conclusion presented in the manuscript.
Conflict of Interest	:	The authors declare no conflict of interest.
Additional Information	:	All information is available for this paper.

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