

Error analysis in algebra learning: Exploring misconceptions and cognitive levels

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Abstract

This research investigates errors and misconceptions among learners in algebraic education by utilizing Koch's error analysis method alongside the Structure of the Observed Learning Outcome (SOLO) taxonomy. The primary aim of the investigation is to discern the kinds of errors and cognitive stages demonstrated by Grade 9 students when engaged in algebraic problem-solving tasks. The studies' outcomes uncover several prevalent error categories, including conjoining, cancellation, and problem-solving errors, indicating deficiencies in conceptual comprehension and procedural execution. Moreover, applying the SOLO taxonomy elucidates learners' diverse levels of understanding, with a majority position within the uni-structural or multi-structural stages. Theoretical implications underscore the necessity for tailored instructional approaches to mitigate learners' obstacles and foster a deeper grasp of algebraic principles. Consequently, this research contributes significantly to the advancement of algebraic pedagogy and provides valuable insights for curriculum enhancement, thereby facilitating improved mathematics learning outcomes.

Keywords: Algebra Learning, Cognitive Levels, Error Analysis, Grade 9 Learners, Misconceptions

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Tiwari and Fatima (2019) explain that primary and secondary school algebra is the foundation for mathematics worldwide and is arguably one of the most important topics in the mathematics curriculum. Without basic algebraic knowledge, learners commit mathematical errors and hold on to misconceptions (Makonye, 2011; Sarımanoglu, 2019). Extending this notion, algebraic errors and misconceptions must be investigated as integral parts of learning to help teachers better design their teaching and learning strategies to address learners' error patterns.

A few studies have begun to address these concerns. For instance, Pournara (2020) examined learners' errors through a diagnostic algebra test in South Africa and examined secondary school learners' algebra performance by comparing them across quintiles. Tiwari and Fatima (2019) examined secondary school students' misconceptions of algebra concepts. While Agustyaningrum et al. (2018) analyzed students' errors in solving abstract algebra, Aygor and Ozdag (2012) examined misconceptions in linear algebra among undergraduate students. However, an additional need remains to critically

analyze errors and misconceptions (Chirume, 2017). Furthermore, Bray and Santagata (2013) articulate the need to develop teaching capacity through knowledge of the types of recognized mathematical errors and misconceptions aiming at their productive use. Bethany (2016) focuses on observing types of mathematical errors and misconceptions to prevent them.

While Mulungye, O'Connor, and Ndethiu (2016, p. 13) explain that “errors are mistakes in the process of solving a mathematical problem algorithmically, procedurally, or by any other method,” they suggest that errors can be explained as a slip-up when mathematical problems are solved algorithmically or procedurally. However, Gardee and Brodie (2015) argue that errors are mistakes that occur systematically and “occur regularly and are pervasive and persistent, often across contexts” and which “occur at a deeper conceptual level than slips” (p. 2). A slip in mathematics is a random blunder that occurs or is produced by carelessness, and these types of errors are easily resolved when identified.

Jointly, an inference regarding errors and misconceptions is that errors, slip-ups, blunders, abnormalities, and false ideas built on incorrect facts are common in the learning of algebra (Aygör & Ozdag, 2012; Baidoo, 2019; Bohlmann et al., 2017; Egodawatte, 2011; Fumador & Agyei, 2018; Iddrisu et al., 2017; Makonye & Fakude, 2016; Mdaka, 2011; Mulungye et al., 2016). Tiwari and Fatima (2019) believe that learners' errors allow teachers to analyze learners' cognitive levels and thinking and, from which, can plan classroom instruction to meet learners' needs.

This study aims to contribute to the existing literature by providing a comprehensive analysis of errors and misconceptions, which can be utilized by educators to design effective teaching approaches and support students in their journey of mastering algebraic concepts, thus the following research questions: What are the underlying cognitive processes contributing to learners' conjoining term errors and how do these errors impact problem-solving, precision, and unpreparedness in algebra among Grade 9 students? How can targeted interventions be developed and implemented to address learners' conjoining term errors and foster early algebra cognition, ultimately improving students' problem-solving skills and conceptual understanding of algebraic concepts?

A review of the literature on the questions first addresses Structures of Observed Learning Outcomes (SOLO) Taxonomy in terms of both nature and scope. To achieve a comprehensive understanding of these aspects, the study adopts Koch's (2014) error analysis and utilizes the SOLO model by Biggs and Collis (2014) to assess Grade 9 learners' proficiency in algebra. To fully address the objective, we employ Koch's (2014) error analysis and Structures of Observed Learning Outcomes (SOLO) model by Biggs and Collis (2014) to classify algebra learning in Grade 9 learners.

The SOLO Taxonomy (Biggs & Collis, 2014) assesses learner cognition or thinking levels regarding algebraic knowledge (Frame, 2018; Lian & Yew, 2012; Na'imah et al., 2018). The SOLO stages are defined as follows:

- *Pre-structural*: The student does not understand the mathematical concept, uses overly simple heuristics, and responds to questions with irrelevant comments.
- *Uni-structural*: The student has minimal understanding of the mathematical concept, only focuses on one relevant aspect of the concept, and usually responds to questions with responses that are shallow, vague, and marginally relevant.
- *Multi-structural*: The student's understanding of the mathematical concept is fragmented, subconcepts are treated independently and disconnectedly, and the student struggles to present or explain the concept.

- *Relational*: The student sees the mathematical concept and its respective subconcepts as an integrated and coherent whole, understands respective patterns, and understands the concept topic from various perspectives.
- *Extended abstract*: The student reconceptualizes the mathematical concept into greater abstraction and generalizations into connected topical areas and may apply the concept in their life.

We can provide parallel descriptors of the SOLO stages based on student responses to questions regarding the mathematical problems:

- pre-structural: the student does not respond to questions, indicating no understanding of the mathematical concept;
- uni-structural: the student provides one response to questions, indicating focus on a singular concept;
- multi-structural: the student provides some responses to questions, indicating a focus on multiple concepts and subconcepts;
- relational: the student readily responds to many questions, indicating recognition of interconnected foci; and
- extended abstract: the student responds to many questions and then asks their own questions connecting mathematics to other realms of mathematics (Fumador & Agyei, 2018).

Addressing the identified research gaps, as reflected in [Table 1](#), will not only contribute to the field of mathematics education but also enable educators to develop targeted interventions and create an inclusive learning environment that empowers all students to excel in mathematics.

Table 1. Exploring Error Patterns and Cognitive Levels in Algebra Problem-Solving

Research Questions	List of Sources	Themes	Directions for New Research	Aims of Research	Sample Gaps in Mathematics Problems
1. Cognitive processes of conjoining term errors and their impact on algebra problem-solving in Grade 9.	Arum et al. (2018)	<ul style="list-style-type: none"> - Cognitive processes of conjoining term errors. - Impact on problem-solving abilities. - Effects on precision in algebraic expressions. - Relationship with unpreparedness in algebra. 	<ul style="list-style-type: none"> - Investigate cognitive mechanisms behind conjoining term errors. - Analyze their effect on problem-solving skills. - Explore the relationship with precision in algebra. - Examine association with unpreparedness 	<ul style="list-style-type: none"> - Develop a comprehensive model of cognitive processes related to conjoining term errors. - Design instructional strategies to address conjoining term errors in algebra. - Investigate the impact of 	<ul style="list-style-type: none"> - Identify common cognitive pitfalls leading to conjoining term errors. - Determine extent of hindrance to problem-solving. - Investigate precision errors from misuse of conjoined terms.

			in algebraic problem-solving.	targeted interventions. - Enhance precision in algebraic expressions through interventions.	- Explore unpreparedness due to conjoining term errors.
2. Development and implementation of interventions to foster early algebra cognition and improve problem-solving skills.	Abdullah et al. (2015)	<ul style="list-style-type: none"> - Development of interventions for conjoining term errors. - Implementation in Grade 9 algebra classes. - Evaluation of intervention effectiveness. - Enhancement of conceptual understanding of algebraic concepts. 	<ul style="list-style-type: none"> - Design evidence-based intervention programs to target conjoining term errors. - Implement interventions and assess their impact. - Measure students' problem-solving skills before and after intervention. - Assess conceptual understanding of algebraic concepts following intervention. 	<ul style="list-style-type: none"> - Investigate the cognitive mechanisms underlying learners' conjoining term errors. - Analyze the effect of conjoining term errors on problem-solving skills. - Explore the relationship between conjoining term errors and precision in algebra. - Examine the association between conjoining term errors and unpreparedness in algebraic problem-solving. 	<ul style="list-style-type: none"> - Develop a comprehensive model of cognitive processes related to conjoining term errors. - Design instructional strategies to address conjoining term errors in algebra. - Investigate the impact of targeted interventions on reducing conjoining term errors. - Enhance students' precision in algebraic expressions through intervention programs.
3. Analysis of learners' errors in algebraic computation and their relationship with foundational mathematical knowledge.	Makonye and Hantibi (2014)	<ul style="list-style-type: none"> - Learners' errors in algebraic computation. - Relationship with foundational mathematical knowledge. 	<ul style="list-style-type: none"> - Analyze learners' errors in algebraic computation. - Examine the connection between errors and foundational 	<ul style="list-style-type: none"> - Develop effective strategies to address errors in algebraic computation. - Enhance students' understanding 	<ul style="list-style-type: none"> - Identify common errors in algebraic computation. - Assess the influence of foundational knowledge on

		- Impact on problem-solving skills.	mathematical knowledge. - Investigate the effect of errors on students' problem-solving abilities.	of foundational mathematical concepts.	error occurrence. - Investigate precision errors resulting from weak foundational understanding. - Identify key factors influencing algebraic proficiency. - Determine the effectiveness of problem-solving strategies. - Investigate the role of conceptual understanding in error prevention.
4. Investigating the impact of problem-solving errors on algebraic proficiency in Grade 9 learners.	Sarimanoglu (2019)	- Impact of problem-solving errors on algebraic proficiency. - Assessment of problem-solving strategies. - Connection with conceptual understanding.	- Explore the relationship between problem-solving errors and algebraic proficiency. - Assess the effectiveness of problem-solving strategies. - Investigate the influence of conceptual understanding on error occurrence.	- Improve algebraic proficiency through targeted interventions. - Enhance problem-solving skills in algebra. - Develop effective instructional methods to prevent problem-solving errors.	- Identify common precision errors in algebraic manipulation. - Assess the impact of precision errors on students' performance. - Investigate the role of procedural knowledge in precision error occurrence.
5. The role of precision errors in algebraic manipulation and their impact on Grade 9 learners' algebraic performance.	Chirume (2017)	- Role of precision errors in algebraic manipulation. - Influence on Grade 9 learners' algebraic performance. - Connection with procedural knowledge.	- Investigate the significance of precision errors in algebraic manipulation. - Analyze their effect on students' algebraic performance. - Examine the relationship with procedural knowledge in algebra.	- Develop effective strategies to reduce precision errors. - Enhance algebraic performance through targeted interventions. - Improve procedural knowledge in algebra.	- Identify common precision errors in algebraic manipulation. - Assess the impact of precision errors on students' performance. - Investigate the role of procedural knowledge in precision error occurrence.
6. Addressing unpreparedness errors in algebraic problem-solving among Grade 9 learners.	Malahlela (2017)	- Unpreparedness errors in algebraic problem-solving.	- Develop interventions to address unpreparedness errors in algebra.	- Enhance algebraic problem-solving skills through targeted interventions.	

- Factors contributing to unpreparedness.	- Identify factors contributing to unpreparedness.	- Improve students' preparedness in algebraic problem-solving.
- Effects on mathematical understanding.	- Assess the impact of unpreparedness errors on mathematical understanding.	- Develop instructional strategies to prevent unpreparedness errors.

Following the extant review of work so far, understanding the nature and implications of problem-solving errors, precision errors, and unpreparedness errors, particularly those related to conjoining terms, is a crucial area of research in mathematics education. Such errors are often driven by a lack of both procedural and conceptual knowledge, leading learners to overlook essential mathematical rules and steps during problem-solving processes. Addressing these challenges is imperative to improve students' algebraic comprehension and problem-solving skills.

Problem-solving errors involving conjoining terms and cancellation errors have been recognized as significant obstacles to students' mathematical progress (Arum et al., 2018; Pournara, 2020; Sarimanoglu, 2019; Sfard, 1991). Past scholarship on learners' errors, misconceptions, and cognitive levels suggests that identifying these may be problematic and, if not corrected early, they could cause increasing distractions to mathematical understanding, performance, and learning (Fumador & Agyei, 2018; Rahim et al., 2015). Thus, educators must understand these dimensions and design robust strategies and tools to analyze and deal with learners' errors and misconceptions.

This study examines Grade 9 learners' errors, misconceptions, and cognitive levels by drawing insights from senior secondary algebra. Koch's (2014) theory of error analysis (also seen in Newman (1977)) focuses on five ways of classifying errors:

- *Careless errors*: Made when learners work carelessly, do not focus, are rushing, or are tired and forgetful.
- *Computation error*: Made by misinterpretation or misuse of operational signs ($\times, \div, +, -$) and incorrect operations using those signs.
- *Problem-solving errors*: Made when learners fail to follow mathematical rules or ignore them because they lack conceptual and procedural knowledge.
- *Precision errors*: Made when a learner writes untidily, is too messy, or drops, misses, or forgets signs, variables, or numbers. In some cases, the learner fails to label or use notation.
- *Unpreparedness errors*: Made when the learner fails to finish the mathematical problem and leaves blank spaces or does not write anything.

While far from an exhaustive list, this investigation draws upon local studies (Kakoma & Makonye, 2010; Malahlela, 2017) and some international studies (Agustyaningrum et al., 2018; Fisher & Frey, 2012) to examine *careless errors*. *Computation errors*, however, are examined through the lens of Makonye and Hantibi (2014), Chirume (2017), and Arum et al. (2018). We also consider the work of Pournara

(2020), Mbewe (2013), Gumpo (2014), Ncube (2016), Koch (2014), and Dhlamini and Kibirige (2014) to examine other ideas, such as *problem-solving errors* involving the conjoining of terms and *cancellation errors*.

Prior research suggests that errors occur unintentionally or unconsciously when solving algebraic problems under various conditions (Luneta & Makonye, 2010), including losing concentration, tiredness, or working quickly. Luneta and Makonye (2010) termed these errors as random or unintended. Some researchers note that *careless errors* are often made by learners incorrectly copying problems or miscopying numbers (Agustyaningrum et al., 2018). Some also recognize that learners commit errors such as not reading the problem in detail and forgetting to write the terms such as $L \subset K$ and $L \neq \emptyset$ in subgroup verification (Bethany, 2016; Brown & Skow, 2016; Elbrink, 2008; Seng, 2010). Bethany (2016) believes *careless errors* are made by working too fast, with learners lacking attention to their problem-solving activity. However, Malahlela (2017) observes that learners may incorrectly write $2x^2 + 7x = 4$ as $2(x + 4)(x - 18)$, and that this does not make mathematical sense and may be a random error.

Additionally, it is unknown why learners mostly cannot realize that they have made a mistake and can fix it independently without help in certain situations. For instance, Bethany (2016) and Agustyaningrum et al. (2018) believe that learners mostly commit mistakes when they fail to read the problem carefully, miscopy the mathematical problem, untidily write, drop signs, fail to follow the procedures, or wrongly write numbers. In contrast, it may be argued that random errors sometimes occur when inexperienced learners miscalculate $2 + 5$ as 10 (i.e., treating addition as multiplication or carelessly failing to notice the addition sign. Makonye (2011) states that random errors often result from learners seemingly forgetting.

Classification of error types often remains problematic. For instance, while Elbrink (2008) believes that *computational errors* can be attributed to carelessness and a lack of attention, Brown and Skow (2016) suggest that carelessness results from learner fatigue or distraction caused by other activities in the classroom. Fisher and Frey (2012) further argue that learners could make careless errors because they are exhausted or unfocused. Chirume (2017) termed these errors mistakes, where learners make mechanical and skill errors (e.g., copying a wrong equation from one of the lines). It is valuable for teachers to understand these errors to best assist students through exact misunderstandings rather than teaching the entire concept again.

Beyond careless errors, researchers have ostensibly been concerned about computation errors (Abdullah et al., 2015; Agustyaningrum et al., 2018; Kakoma & Makonye, 2010; Makonye & Hantibi, 2014; Ncube, 2016). *Computation errors* happen when learners fail to operate signs such as addition, subtraction, division, and multiplication. Elbrink (2008) denotes these errors as mis-addition, mis-subtracting, mis-multiplying, and mis-dividing. Nevertheless, Ncube (2016) suggests such errors may include incorrect use of signs, where the error lies in operational signs and integers. For example, it has been revealed that in some cases, learners cannot expand $-p(q - 3)$ and instead revealed the solution as $-pq - 3p$. While Abdullah et al. (2015) and Makonye and Hantibi (2014) termed these types of errors as transformation errors, noting that they involve the confusion of operations and signs (e.g., $+$, $-$, \times , \div) leading to errors such as $-3 \times -5 = -8$. However, this does not adequately explain the phenomenon. For instance, it is unclear why learners confuse signs and operations (Kakoma & Makonye, 2010). Abdullah et al. (2015) wonder why these errors often appear in learners who already know the correct solution but fail to answer correctly and acknowledge algebraic operations.

Even though there has been considerable research suggesting that learners tend to lack basic primary school mathematics knowledge (e.g., failure to correctly perform addition, multiplication, division, and subtraction and correctly use signs and symbols), it still does not explain the relationship between these operations and why learners fail to distinguish the relationship between these operations (e.g., a positive sign versus the addition operation or a negative sign versus the subtraction operation) (Chirume, 2017). Indeed, Chirume (2017) suggests that the knowledge gap caused by different kinds of errors may result from a lack of awareness of basic facts, properties, and principles due to other types of errors (e.g., *problem-solving*, *precision*, and *unpreparedness errors*).

Many believe that problem-solving errors involve conjoining terms and cancellation errors (Arum et al., 2018; Dhlamini & Kibirige, 2014; Gumpo, 2014; Ncube, 2016; Pournara, 2020; Sfard, 1991) usually caused by a lack of procedural and conceptual knowledge. Learners ignore mathematical rules because they lack conceptual understanding of the mathematics and problem-solving steps taken (Sfard, 1991). Arum et al. (2018) notice that some learners cannot self-assess their understanding and work and must develop problem-solving skills to address algebraic rules and procedures correctly.

However, resolving such challenges remains a concern. Researchers such as Dhlamini and Kibirige (2014), Gumpo (2014), and Ncube (2016) have identified types of errors regarding solving algebraic equations (e.g., conjoining terms or mis-using signs to join terms, including variables or numbers, operation signs, and errors associated with commutative and distributive properties). Recent evidence by Ung et al. (2019) found the following: When a learner was given: $f(x) = 2x^2 - 6x$ and asked to find x if $f(x) = 4x$. The learner responded, $2x^2 - 6x = 0 - 4x = 0$. Another learner responded, $2x^2 - 6x = 0 = -4x^3$.

In a parallel study, Pournara (2020) identified more granular error types such as: like and unlike term errors, negative sign errors, and negative pre-multiplier as causes of such errors. As operational signs are introduced in the lower grades, errors involving like and unlike terms develop from poor basic knowledge acquired in the lower grades. Anchored on Sfard's (1991) assertion, Pournara (2020) opines that conjoining errors must be corrected earlier because it is key to early algebra cognition, as such in the current study.

METHOD

Guided by the research objective, the methodology was principally interpretivism and occasionally triangulated (mixed method approach). This mixed-method research (MMR) includes a combination of two approaches that are quantitative and qualitative (Creswell & Clark, 2007). The MMR was applied to this research because the researchers investigated contextual analysis of errors and misconceptions in learning algebra concurrently with a descriptive survey design to respond to the type of errors and misconceptions that occurred. In such an instance, the data generation and collection instrument for investigating the types of errors and misconceptions included learners' written tests counting 50 marks. Based on the examples used in the literature review, the four questions in the test comprised algebraic expressions and equations, fractions, graphs, tables, word problems, and the concept of ratio, height, length, and area.

The study's sample comprised one hundred (100) Grade 9 learners selected from five schools from two circuits, Umlalazi and Mtunzini, in King Cetshwayo district, for the test. Data were analyzed using descriptive statistics for types of errors and misconceptions in Grade 9 learners when learning algebra. The data were analyzed following Koch's error analysis rules as directed by the theory section.



Similarly, learners' cognitive levels in terms of the identified types of errors were analyzed according to the SOLO model. Twenty learners in each of the five sampled schools were selected to respond to types of errors and misconceptions when learning algebra. Mathematics teachers at the schools validated the test. Ethical clearance from the University of Maths (pen-named) and the Kwa Zulu-Natal Department of Education granted a clearance certificate to the researchers.

This study explored learners' cognitive levels and types of errors and misconceptions by drawing insights from senior secondary algebra through the employment of Koch's error analysis and SOLO model by Biggs and Collis to classify algebra learning in Grade 9 learners. Accordingly, the current section starts with the analysis of Koch's error analysis.

The study utilized a mixed-method approach and incorporated both quantitative and qualitative methodologies to exploring conjoining term errors in algebra among Grade 9 students. Quantitatively, standardized tests were administered to evaluate algebra proficiency and identify error patterns, with subsequent descriptive analysis to establish with factors. Qualitatively, semi-structured interviews with students, complemented by classroom observations, delved into the thought processes and perceptions regarding algebraic concepts. Meanwhile the thematic analysis the qualitative data unearthed recurring themes in students' understanding and misconceptions.

The analysis technique utilized descriptive statistics for the quantitative data, focusing on summarizing and presenting this data in a comprehensible format. This approach involved percentages. This was chosen for its suitability in aligning with our research question to identify and describe error types and misconceptions in algebra among Grade 9 learners. This option also allowed for an initial, rigorous assessment of the data, appropriating the exploratory phase of the research, without the need for more complex inferential statistics.

This study is the authors' original work, which has not been previously published elsewhere. Ethical approval was sought and granted by the ¹XXX Faculty of Education's research ethics committee, which helped the researcher adhere to the norms and practices inherent to ensuring participants were protected. Permission was sought to conduct the research from the participating schools. The informed consent form signed by the participant accounted for strict adherence to the protection of personal information while confidentiality, privacy, and anonymity were observed at all times.

RESULTS AND DISCUSSION

The primary focus of this study was to investigate learners' cognitive levels and types of errors and misconceptions in senior secondary algebra, specifically among Grade 9 learners. This study explored learners' cognitive levels and types of errors and misconceptions by drawing insights from senior secondary algebra through the employment of Koch's error analysis and SOLO model by Biggs and Collis to classify algebra learning in Grade 9 learners. Accordingly, the current section starts with the analysis of Koch's error analysis.

Figure 1 reveals the distribution of error types, with *problem-solving errors* being the most common errors found in the study. As the literature indicates, such high levels of *problem-solving errors* lead to different types of other errors, such as *cancellation errors* and the conjoining of terms (Fumador & Agyei, 2018; Tiwari, 2019). The *problem-solving errors* were followed by *unpreparedness errors*, at 73%, which are errors committed by learners when they fail to complete mathematical problems or leave blank

¹ For review purpose and to conceal identifiable sources

responses, showing a lack of procedural and conceptual knowledge. However, 49% committed computational errors due to failure to perform operations (+/- and \div/\times) or performing them improperly.

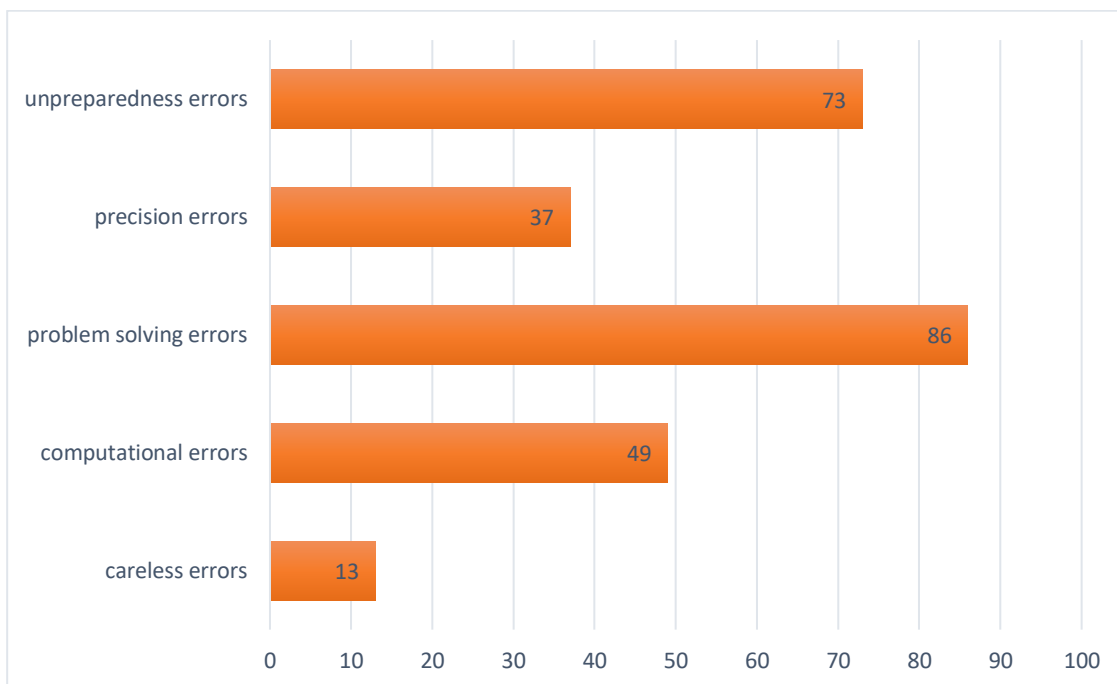


Figure 1. Distribution of types of errors

The literature review classified various errors, including *problem-solving errors* (Gumpo, 2014; Mashazi, 2014). *Cancellation errors* are also explained as cancelling similar variables without following mathematical rules or a reason (Makonye & Fakude, 2016). An example includes the addition of numerators and denominators when learners were dealing with fractions or adding numerators and denominators without following mathematical rules. However, Figure 3 reflects the joining of the term by ignoring the sign between the terms. In further examining the student's work, the sign (+) was used to join terms, e.g., $2x^2 + 3 = 5x^2$. Additionally, Figure 2 was also drawn without following the mathematical rules. These *problem-solving errors* led to conjoining errors (i.e., joining terms and ignoring signs or using them as a joiner of terms) and cancellation errors, as reflected in Figure 3 (the equation associated with the graph).

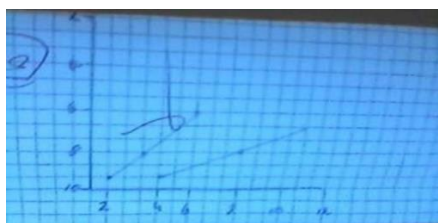


Figure 2. Learner response showing problem-solving error

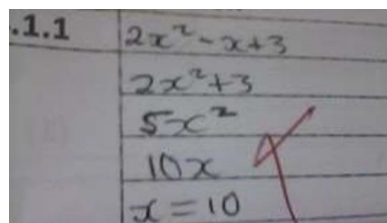


Figure 3. Conjoining of error

Figure 4. Showing the cancellation error

Figure 5. Showing Addition of numerators and addition of denominators

Figure 6. Showing computation errors

Figure 7. Careless error and precision error

Figure 8. Precision and careless errors

Figure 9. An unpreparedness error

On the other hand, computational errors were also prevalent, with learners needing to be more accurate with addition, multiplication, subtraction, and division operations and signs, as shown in Figure 6. Furthermore, Figures 2–9 show the contextual analysis algebra tasks via Kock analysis.

As identified in the literature, conjoining errors related to problem-solving (Gumpo, 2014; Mashazi, 2014) were also found in learner's scripts (e.g., where $2x^2 + 3 = 5x^2 = 10x$, $x + 4 = 4x$, and $x^2 - 16 = 16x$). In these, learners notably revealed both a lack of conceptual and procedural knowledge and ignored algebraic rules through conjoining terms, ignoring the operational sign, and conjoining variables with a number (e.g., $4x$, which differs from $4 + x$).

Mashazi (2014) and Gumpo (2014) documented errors when learners ignore signs or a letter in solving an expression or equation. Their studies discovered that errors occur if a learner unsuccessfully links new knowledge with existing one in learning algebra. Learners also made errors by skipping a letter or a variable. In response, Malahlela (2017) maintains that new knowledge delivered to learners from teachers in class depends on previously held cognitive information. However, it also depends on how knowledge is delivered and how old knowledge integrates new knowledge in the learner's mind. For instance, Alshwaikh and Adler (2017) identified learner errors in simplifying (e.g., $(x + 2)(x + 4) = 2x + 6$), and Jacobs et al. (2014) refer to a "like term error" such as $2x + y = 2xy$.

Other learners in this study made cancellation errors. As reflected in the literature review, Makonye and Fakude (2016) claim this is due to cancelling without understanding. As with some learners, it was observed that similar variables in the problem were canceled while ignoring the appropriate algebraic rules. According to Koch's error analysis, this error is classified as a *problem-solving error*, where the learner cannot follow the rules to solve the problem. In summary and using a similar notion but different terminologies, researchers (e.g., Dhlamini & Kibirige, 2014; Makonye & Hantibi, 2014) classify these

errors as conceptual errors, a notion supported by others (e.g., Brown & Skow, 2016; Fisher & Frey, 2012)

Analysis, according to the SOLO Taxonomy, produced a mix of ideas. For instance, one of the learners needed to follow a proper mathematical rule in performing mathematical operations on fractions. This learner chose to add numerators and denominators, resulting from a *problem-solving error*. Thus, concept errors may result from misconceptions or faulty understanding of the underlying principles and ideas connected to the mathematical problem. On the other hand, error analysis revealed that these errors are built on learners' failure to follow proper mathematical rules. In contrast, others solve problems on their own and ignore rules because they lack conceptual and procedural knowledge, leading to performing incorrect mathematical steps.

Other considered errors were *computational errors*, *careless errors*, *precision errors*, and *unpreparedness errors*. For instance, learners committed *computational errors* where they misinterpreted operational signs, e.g., one of the learners confused a multiplication sign with a subtraction sign. In one case, the learner failed to solve an equation containing a fraction, but they did indicate some conceptual knowledge by cross multiplying. However, they inexplicably inserted negative signs: $5x - (3x - 3) = 5 - (7x - 7)$. This idea applies to the SOLO uni-structural stage, as the learner considered only one concept in the mathematics. *Computation errors* usually occur when learners make mistakes by misinterpreting or misusing operational signs ($\times, \div, +, -$) and operating with those signs incorrectly.

One of the learners made a careless mistake when they did not follow the correct mathematical rule when writing down the expression; instead of substitution, they either forgot the x variable or dropped it by mistake and inserted an exponent of 2 in the second term. In this case, the learner solved their mathematical problem carelessly and ignored the correct method. The learner's work and responses revealed that they considered only one mathematical concept (SOLO's uni-structural stage) (Frame, 2018). For instance, the learner wrote down the correct expression but failed to perform the appropriate substitution and then committed other errors. The learner's work and responses focus on multiple (mis)concepts in the problem that (mis)informs them to continue committing more errors. Despite foundational misunderstandings, the learner's work and communication revealed they were positioned in the multi-structural SOLO stage, as they wrote the correct formula and substituted the correct values. The error occurred when the learner forgot to place the bracket and computed it wrongly. The learner revealed many ideas that symbolize conceptualization.

Some learners committed *precision errors* because messy work resulted in dropping parenthesis, the sign of a number, or the variable itself. For instance, some learners in this study committed *precision errors* by writing the mathematical problem untidily, failing to place the multiplication sign or brackets to show multiplication, and canceling without following appropriate algebraic rules. While failing to place brackets in the expression was careless and led to the learner not multiplying all appropriate terms, this mistake could be realized and rectified later. Additionally, while some learners revealed multi-structural reasoning, they did so despite making errors along the way.

Notably, and in summary, a few participating learners were positioned at the pre-structural SOLO stage. Most study participants were recognized in the uni-structural or multi-structural stage, and none were deemed in the relational stage.

While one might hypothesize that types of errors committed (i.e., careless errors, computation error, problem-solving errors, precision errors, and unpreparedness errors) regarding solving algebra problems may be correlated to various SOLO stages (i.e., pre-structural, uni-structural, multi-structural, relational, and extended abstract), however, this study may demonstrate no correlation whatsoever

between the two dimensions. Indeed, this study found that students could perform particular errors whether positioned in the uni-structural or multi-structural stages (the only stages significantly represented among the study participants). Indeed, in the problem-solving process, errors were often followed by continued work that may have demonstrated understanding and even higher SOLO stages. Conversely, there was little predictability of error types predicated on SOLO stages. Indeed, only when a learner halted at some point in the problem-solving process and failed to respond further to questions was there a cessation in the flow of the work-related data to best consider into which SOLO stage they should be positioned. Therefore, (A) a student's levels of algebraic learning (concerning the SOLO taxonomy) cannot be determined by the error types they commit in their algebraic problem solving, and (B) given a student's placement in the SOLO taxonomy cannot predict what types of errors they may perform.

CONCLUSION

While there are several implications from the study's results and the literature, one principal implication is that conjoining errors are sourced from a lack of knowledge in solving algebraic equations. It is built on conjoining errors related to incomplete simplification, where learners do not complete or follow all the steps. Instead, they devise a shortcut and an unacceptable method for calculating the algebraic expression. Rather than simply investigating student algebraic errors, one must study their rationale more deeply for the shortcuts they attempt and why they perform their steps. For instance, regarding a recognized problem-solving error, Makonye and Fakude (2016) show that learners cancel terms without following appropriate rules and other conjoin terms. Indeed, in this study, some learners did not complete all the required steps and improperly cancelled similar terms. Thus, again, there is a greater need to discern why a student performs particular actions more than simply determining and listing the errors they commit.

Altogether, basic algebraic skill needs to be revisited by both teachers and learners, which will assist learners in building a solid mathematics foundation. The consensus is that learners need to understand the use of operational signs and the difference between monomials, binomials, trinomials, and polynomials. Consequently, relearning algebraic rules and concepts such as "term" is recommended, so learners follow proper mathematical rules when solving algebra problems.

While it was found that most study participants were at the SOLO uni-structural and multi-structural stages and performed one or more or various combinations of the errors (i.e., careless errors, computation errors, problem-solving errors, precision errors, and unpreparedness errors), it may be valuable to consider studies that blend different frameworks to deduce additional findings. This could revolutionize future mathematics education research.

This investigation motivates future studies regarding secondary-level students' conceptual errors encountered in algebra problem solving and consequently provides a foundation for developing teaching capacity and making productive use of mathematics errors. Altogether, two main conclusions are made: First, an analytical approach could identify errors and misconceptions. Specific aspects of learners' understanding, and particular needs could be addressed upon error identification. Second, students and teachers should be aware of computation, problem-solving, and precision errors and develop new strategies to avoid these error types. The teacher must prepare learners to solve algebra problems while avoiding the numerous errors discussed above.

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REFERENCES

- Abdullah, A. H., Liyana, N., Abidin, Z., & Ali, M. (2015). Analysis of students' errors in solving higher order thinking skills (HOTS) problems for the topic of fractions. *Asian Social Science*, 11(21), 58-78. <http://dx.doi.org/10.5539/ass.v11n21p133>
- Agustyaningrum, N., Abadi, A. M., Sari, R. N., & Mahmudi, A. (2018). An analysis of students' error in solving abstract algebra tasks. *Journal of Physics: Conference Series*, 1097(1), 012118. <https://doi.org/10.1088/1742-6596/1097/1/012118>
- Alshwaikh, J., & Adler, J. (2017). Researchers and teachers as learners in Lesson Study. In M. K. Mhlolo, S. N. Matoti, & B. Fredericks (Eds.), *SAARMSTE Book of Long Papers* (pp. 2-14). https://www.researchgate.net/profile/Jehad-Alshwaikh/publication/312083661_Researchers_and_teachers_as_learners_in_Lesson_Study/links/58d527f2458515337857a616/Researchers-and-teachers-as-learners-in-Lesson-Study.pdf
- Arum, D. P., Kusmayadi, T. A., & Pramudya, I. (2018). Students' logical-mathematical intelligence profile. *Journal of Physics: Conference Series*, 1008(1), 012071. <https://doi.org/10.1088/1742-6596/1008/1/012071>
- Aygor, N., & Ozdag, H. (2012). Misconceptions in linear algebra: the case of undergraduate students. *Procedia - Social and Behavioral Sciences*, 46, 2989-2994. <https://doi.org/https://doi.org/10.1016/j.sbspro.2012.05.602>
- Baidoo, J. (2019). Dealing with grade 10 learners' misconceptions and errors when simplifying algebraic fractions. *Journal of Emerging Trends in Educational Research and Policy Studies*, 10(1), 47-55. <https://doi.org/doi:10.10520/EJC-17aa794007>
- Bethany, Y. (2016). 3 Types of math errors and how to prevent them. <https://mathgeekmama.com/types-of-math-errors/>
- Biggs, J. B., & Collis, K. F. (2014). *Evaluating the quality of learning: The SOLO taxonomy (Structure of the Observed Learning Outcome)*. Academic Press.
- Bohlmann, C. A., Prince, R. N., & Deacon, A. (2017). Mathematical errors made by high performing candidates writing the National Benchmark Tests. *Pythagoras*, 38(1), 1-10. <https://doi.org/doi:10.4102/pythagoras.v38i1.292>
- Bray, & Santagata, W. (2013). How to leverage the potential of mathematical errors. *Teaching Children Mathematics*, 19(7), 424-431. <https://doi.org/10.5951/teacchilmath.19.7.0424>

- Brown, J., & Skow, K. (2016). *Mathematics: Identifying and addressing student errors*. IRIS Center. https://iris.peabody.vanderbilt.edu/wp-content/uploads/pdf_case_studies/ics_matherr.pdf
- Chirume, S. (2017). A critical analysis of errors made by rural and urban students in 'A' level mathematics paper 1 (4008/1) in Shurugwi and Gweru Districts, Zimbabwe. *Asian Journal of Education and e-Learning*, 5(2), 63-73. <https://pdfs.semanticscholar.org/aecb/4bd1a018ab169e60af4403032352cfe36897.pdf>
- Creswell, J. W., & Clark, V. L. P. (2007). *Designing and conducting mixed methods research*. Sage Publications, Inc.
- Dhlamini, Z. B., & Kibirige, I. (2014). Grade 9 learners' errors and misconceptions in addition of fractions. *Mediterranean Journal of Social Sciences*, 5(8), 236-256. <https://doi.org/10.5901/mjss.2014.v5n8p236>
- Egodawatte, G. (2011). *Secondary school students' misconceptions in algebra*. University of Toronto, Toronto.
- Elbrink, M. K. (2008). *Analyzing and addressing common mathematical errors in secondary education*. Ball State University, Cardinal Scholar. <http://liblink.bsu.edu/catkey/1365327>
- Fisher, D., & Frey, N. (2012). Making time for feedback. *Educational Leadership*, 70(1), 42-47. <https://www.ascd.org/el/articles/making-time-for-feedback>
- Frame, R. (2018). *What is SOLO taxonomy?* <https://edu.rsc.org/ideas/what-is-solo-taxonomy/3008902.article>
- Fumador, E. S., & Agyei, D. D. (2018). *Students' errors and misconceptions in algebra: Exploring the impact of remedy using diagnostic conflict and conventional teaching approaches*. U. o. C. C. I. Repository. <http://hdl.handle.net/123456789/8054>
- Gardee, A., & Brodie, K. (2015). A teacher's engagement with learner errors in her Grade 9 mathematics classroom : original research. *Pythagoras*, 36(2), 1-9. <https://doi.org/doi:10.4102/pythagoras.v36i2.293>
- Gumpo, L. (2014). *Grade 9 learners' strategies and errors in solving arithmetic and algebraic linear equations* (Doctoral dissertation, University of the Witwatersrand, Faculty of Science, School of Science Education).
- Iddrisu, M. M., Abukari, A., & Boakye, S. (2017). Some common misconstructions and misinterpretations in basic algebra: A case of students of university for development studies and Navrongo senior high school in Ghana. *British Journal of Education*, 5(9), 22-35.
- Jacobs, M., Mhokure, D., Fray, R. L., Holtman, L., & Julie, C. (2014). Item difficulty analysis of a high-stakes mathematics examination using Rasch analysis. *Pythagoras*, 35(1), 1-7. <https://doi.org/10.4102/pythagoras.v35i1.220>
- Kakoma, L., & Makonye, P. J. (2010). Learners and misconceptions in elementary analysis: A case study of a Grade 12 class in South Africa. *Acta Didactica Napocensia*, 3(3), 35-46. <https://files.eric.ed.gov/fulltext/EJ1056125.pdf>
- Koch, M. A. (2014). *The relationship between peer coaching, collaboration and collegiality, teacher effectiveness and leadership* (Doctoral dissertation, Walden University).

- Lian, L. H., & Yew, W. T. (2012). Assessing algebraic solving ability: A theoretical framework. *International Education Studies*, 5(6), 177-188. <http://dx.doi.org/10.5539/ies.v5n6p177>
- Luneta, K., & Makonye, P. J. (2010). Learner Errors and Misconceptions in Elementary Analysis: A Case Study of a Grade 12 Class in South Africa. *Acta Didactica Napocensia*, 3(3), 35-46.
- Makonye, J. P. (2011). Learner errors in introductory differentiation tasks: A study of learner misconceptions in the National Senior Certificate examinations. *University of Johannesburg, Auckland Park, South Africa*.
- Makonye, J. P., & Fakude, J. (2016). A study of errors and misconceptions in the learning of addition and subtraction of directed numbers in grade 8. *SAGE Open*, 6(4), 2158244016671375. <https://doi.org/2158244016671375>
- Makonye, J., & Hantibi, N. (2014). Exploration of grade 9 learners' errors on operations with directed numbers. *Mediterranean Journal of Social Sciences*, 5(20), 1564-1572. <https://doi.org/10.5901/mjss.2014.v5n20p1564>
- Malahlela, M. V. (2017). *Using errors and misconceptions as a resource to teach functions to grade 11 learners*. University of the Witwatersrand, Johannesburg, South Africa.
- Mashazi, S. (2014). Learners' explanations of the errors they make in introductory algebra. In U. o. Witwatersrand (Ed.), *Wits Maths Connect secondary project, School of education*.
- Mbewe, T. L. (2013). *Misconceptions and errors in Algebra at grade 11 levels: The case of two selected secondary schools in Petauke District Zambia*.
- Mdaka, B. R. (2011). *Learners' errors and misconceptions associated with common fractions*. University of Johannesburg (South Africa).
- Mulungye, M. M., O'Connor, M., & Ndethiu, S. (2016). Sources of student errors and misconceptions in algebra and effectiveness of classroom practice remediation in Machakos County-Kenya. *Journal of Education and Practice*, 7(10), 31-33. <https://files.eric.ed.gov/fulltext/EJ1099568.pdf>
- Na'imah, S., Sulandra, M., & Rahardi, R. (2018). Kemampuan siswa field dependent level multistructural dalam menyelesaikan soal pythagoras dan pemberian scaffolding [Ability of multistructural level field dependent students in solving pythagorean problems and scaffolding]. *Jurnal Pendidikan: Teori, Penelitian, dan Pengembangan*, 3(6), 788-798. <https://doi.org/10.17977/jptpp.v3i6.11234>
- Ncube, M. (2016). *Analysis of errors made by learners in simplifying algebraic expressions at grade 9 level*. University of South Africa. <http://hdl.handle.net/10500/23199>
- Newman, M. A. (1977). An analysis of sixth-grade pupils' errors on written mathematical tasks. In M. A. C. J. Foyster (Ed.), *Research in Mathematics Education in Australia* (Vol. 2, pp. 269-287). Swinburne College Press.
- Pournara, C. (2020). Grade 9 learners' algebra performance: Comparisons across quintiles, insights from errors and curriculum implications. *South African Journal of Science*, 116, 1-7. http://www.scielo.org.za/scielo.php?script=sci_arttext&pid=S0038-23532020000600011&nrm=iso
- Rahim, R. A., Noor, N. M., & Zaid, N. M. (2015). Meta-analysis on element of cognitive conflict strategies with a focus on multimedia learning material development. *International Education Studies*, 8(13), 73-78. <https://doi.org/10.5539/ies.v8n13p73>



- Sarimanoglu, N. U. (2019). The investigation of middle school students' misconceptions about algebraic equations. *Studies in Educational Research and Development*, 3(1), 1-22.
- Seng, L. K. (2010). An error analysis of Form 2 (Grade 7) students in simplifying algebraic expressions: A descriptive study. *Electronic Journal of Research in Education Psychology*, 8(20), 139-162. <https://doi.org/https://doi.org/10.25115/ejrep.v8i20.1398>
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics*, 22(1), 1-36. <https://doi.org/10.1007/BF00302715>
- Tiwari, C., & Fatima, R. (2019). Secondary school students' misconceptions in algebra concept. *Mahatma Gandhi Central University Journal of Social Sciences*, 1(1), 22-33. <https://mgcub.ac.in/pdf/journals/mgcujs 1-1 april-september 2019-32-43.pdf>
- Ung, H. Q., Phan, M. K., Nguyen, H. T., & Nakagawa, M. (2019). Strategy and tools for collecting and annotating handwritten descriptive answers for developing automatic and semi-automatic marking-an initial effort to math. In *2019 International Conference on Document Analysis and Recognition Workshops (ICDARW)* (Vol. 2, pp. 13-18). IEEE.

