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How mathematics teachers' special knowledge changing: A case study in the Professional Teacher Education program

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Abstract

The COVID-19 pandemic has catalyzed the widespread adoption of distance learning, necessitating a comprehensive understanding of how teacher knowledge evolves within the context of Teacher Professional Education (TPE) programs. Some research endeavors may employ evaluation methodologies that inadequately capture the nuanced dimensions of knowledge, highlighting the necessity for greater incorporation of evidence-based approaches in the formulation and assessment of teacher development initiatives. This study employs a qualitative methodology to explore the evolution of Mathematics Teachers' Specialized Knowledge (MTSK) within the framework of a TPE program. Data were gathered through the engagement of three educators in the preparation and execution of lessons on permutation and combination via a Learning Management System (LMS), coupled with in-depth interviews. The findings underscore the TPE program's role in fostering collaborative learning environments through participation in online educational communities. Teachers are shown to be increasingly integrating technology into their pedagogical practices, albeit with varying degrees of proficiency. The presence of Professional Learning Communities (PLCs) is identified as instrumental in supporting educators in refining instructional strategies to enhance teaching effectiveness. Nevertheless, there remains a subset of teachers necessitating more profound and more comprehensive content knowledge about their subject matter. These insights emphasize the importance of designing TPE programs that offer sustained and adaptable professional development opportunities to facilitate continuous growth among educators. Consequently, it is recommended that online learning communities geared toward addressing the diverse learning requirements of students be established, thereby aiding in the identification and resolution of pedagogical challenges.

Keywords: Mathematics Teachers Specialized Knowledge, Permutation and Combination, Professionalism Development, Teacher Profession Education

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Since 2017, the Indonesian government has pursued reforms in the teacher training system by implementing Teacher Professional Education (TPE) programs. These programs aim to equip graduates with comprehensive teacher competencies following national education standards (Permendikbudristek, 2022). As part of this initiative, the TPE program offers a range of professional development opportunities explicitly tailored for mathematics educators (Kusumah & Nurhasanah, 2017). Despite the program's positive impact on enhancing teacher professionalism, significant challenges persist in Indonesia's teaching profession, particularly concerning the establishment and execution of professional teacher standards during both the selection process and the fulfillment of teaching responsibilities (Revina et al., 2020; Yusrina et al., 2022). Therefore, conducting investigations into teachers' knowledge and the advancement of mathematics instruction in classrooms assumes paramount importance for assessing the efficacy of the TPE program, particularly within the realm of mathematics education.

Mathematics Teachers' Specialized Knowledge (MTSK) encompasses the specialized understanding of mathematics educators possess, serving as a hallmark of their proficiency in advancing mathematics education. Carrillo-Yañez et al. (2018) conceptualize MTSK as a teacher framework, guiding the integration of mathematical content with pedagogical practices. This model delineates teachers' engagement in lesson planning, peer collaboration, lesson delivery, and post-teaching reflection. Furthermore, Ivars et al. (2017) also elucidate MTSK as the accumulation of knowledge acquired through professional development endeavors intertwined with teachers' recollection and contemplation on teaching specific topics. Cultivating MTSK among teachers is paramount, especially as they strive to enhance their grasp of mathematics content for classroom instruction and learning (Scheiner et al., 2019). Consequently, continuous professional development activities are essential for mathematics teachers to augment their expertise (Kusumah & Nurhasanah, 2017). Moreover, the Department of Education and Training (2022) underscores the significance of professional learning communities in addressing students' learning requirements, facilitating ongoing professional growth, and refining teaching practices.

Numerous studies have underscored the integral connection between the proficiency of (prospective) mathematics teachers in teaching and various facets of their professional competence (Kirwan et al., 2023; König et al., 2021; Lavidas et al., 2022; Nzaramyimana & Umugiraneza, 2023). This perspective posits that teachers must possess not only a profound comprehension of mathematical content but also specific pedagogical knowledge that amalgamates mathematical concepts with an understanding of students' needs (Kirwan et al., 2023). Additionally, it emphasizes the importance of grasping teaching quality and its impact on students' mathematics achievement (König et al., 2021), as well as the adept utilization of technology in instruction (Lavidas et al., 2022; Nzaramyimana & Umugiraneza, 2023).

Furthermore, research highlights the significance of teachers' knowledge in teaching mathematics regarding the professional development programs they engage in. Some studies indicate that educators participating in programs spanning several months encounter difficulties in attending to student errors and implementing effective problem-solving strategies (Matitaputty et al., 2022; Yunianto et al., 2021). Conversely, a five-year longitudinal professional development program has been shown to bolster teachers' confidence and fortify their pedagogical approach to mathematics education (Auslander et al., 2023). Nevertheless, teachers continue to face challenges in effectively imparting mathematical concepts to their students.

Assessing teachers' specialized knowledge in mathematics presents challenges, especially concerning complex topics such as combinatorics. Combinatorics holds a pivotal position within the mathematics curriculum, underpinning principles of calculation, computation, and probability (Lockwood et al., 2020; NCTM, 2000; Sandefur et al., 2022; Soto et al., 2022). Integrated into the curriculum, combinatorics, particularly permutation and combination, constitutes a crucial component of the probability and statistics domain. These concepts are instrumental in facilitating students' comprehension of conditional probability and mutually independent events (Kemendikbudristek BSKAP, 2022).

Practical instruction of combinatorics, specifically permutation and combination, necessitates teachers' profound comprehension of the subject matter alongside adept pedagogical skills. Moreover, the content of these topics serves a dual purpose: not only does it impart specific mathematical concepts, but it also cultivates problem-solving strategies and nurtures critical mathematical processes such as generalization and recursive thinking among students (Kapur, 1970; Lamanna et al., 2022b; Lockwood et al., 2015; 2020; Sandefur et al., 2022).

Contrary to expectations, several studies have highlighted students' persistent challenges when grappling with permutation and combination problems, often stemming from deficiencies in teachers' content mastery and pedagogical approaches. Matitaputty et al. (2022) investigated educators enrolled in the TPE program and identified instances where teachers struggled to identify and address students' errors in permutation and combination problem-solving. Moreover, instructional strategies predominantly focused on formulaic explanations, as observed in the research conducted by Lamanna et al. (2022a) exacerbating students' difficulties in solving combination problems.

To counteract these challenges, an alternative approach to teaching combinatorial concepts is warranted, transcending procedural applications and delving into fostering deeper conceptual understanding among students' thinking in solving problems (Lockwood et al., 2020). Teachers are encouraged to facilitate experiences that prompt students to engage with mathematical ideas in a manner aligned with their cognitive processes (Soto et al., 2022). Leveraging computational contexts can offer valuable insights into the significance of sequence within permutation and combination materials (Lockwood, 2019). Furthermore, Lockwood et al. (2020) advocate further exploring teachers' knowledge influencing professional development in combinatorics to craft elementary problems that stimulate students' nuanced and profound thinking.

In tandem with teachers' engagement in professional development initiatives, Mathematics Teachers' Specialized Knowledge (MTSK) development is significantly influenced by their participation in Professional Learning Communities (PLCs). These communities serve as forums where educators convene to exchange insights, share experiences, and engage in reflective dialogue regarding their pedagogical practices, particularly in the domain of mathematics instruction (Little, 2020). The collaborative dynamics within PLCs play a pivotal role in shaping teachers' expertise in mathematics, as these communities serve as catalysts for the cultivation of the MTSK (Sigurdardóttir, 2010). Through active participation in PLCs, teachers can enhance their professional knowledge and contribute to improved student learning outcomes (Vescio et al., 2007). Furthermore, researchers seek to illuminate the nuances of teachers' MTSK to comprehensively understand their proficiency in teaching mathematics, particularly concerning permutation and combination materials. Leveraging the interpersonal connections established within PLCs among peers, supervisors, and student teachers, researchers endeavor to identify the challenges and obstacles educators encounter in teaching mathematics.

Chauraya and Brodie (2017) demonstrated the effective utilization of Professional Learning Communities (PLCs) to intervene in teachers' lesson planning, thereby enhancing students' conceptual understanding of mathematical concepts such as rational numbers and ratios. Their study underscored the pivotal role of PLCs in facilitating shifts in teaching methodologies among educators and elucidated the resultant changes in teachers' knowledge acquisition. Conversely, Harvey and Teledahl (2022) identified notable transformations among mathematics teachers engaged in PLCs, including shifts in collaboration norms, heightened comprehension of mathematical principles and their pedagogical application, and enhanced proficiency in designing and executing mathematics instruction. However, despite these advancements, the precise role of PLCs within the learning process still needs to be elucidated. Existing research primarily focuses on establishing PLCs rather than harnessing their full potential as platforms for fostering MTSK within the domains of cognition and interaction. Moreover, a comprehensive exploration of the impact of PLCs, specifically in the context of permutations and combinations, remains warranted.

Campbell and Lee (2017) concluded that mathematics teachers engaging in PLCs primarily centered their discussions on mathematical content, thereby providing fertile ground for developing pedagogical knowledge. Additionally, Matitaputty et al. (2022) successfully evaluated teachers' content and pedagogical proficiency in teaching permutations and combinations by analyzing student errors. Nevertheless, further evidence is required to address teachers' knowledge gaps in this area comprehensively. In light of these findings, this study advocates for assessing the role of PLCs within the TPE program, particularly concerning evaluating teachers' MTSK in teaching permutations and combinations. This endeavor aims to shed light on the efficacy of PLCs as catalysts for enhancing teachers' pedagogical approaches and fostering a deeper understanding of mathematical concepts among both teachers and students.

Based on an analysis of various theoretical perspectives, we have identified significant potential in structuring the TPE program to incorporate PLCs. PLCs offer a structured framework through which teachers can cultivate their knowledge and professionalism within their field. Given the contextual nature of permutations and combinations, variations in teacher knowledge underscore the relevance of PLCs within the TPE program. Therefore, this paper focuses on teachers' activities in designing and implementing permutations and combinations of lessons within the TPE program, aiming to assess the program's efficacy and ensure maximal benefit for participating teachers.

The subsequent section will elaborate on the methodology employed, delineating the research design and highlighting four pivotal stages integral to discerning changes in MTSK throughout the TPE program. The following section will present the results, showcasing the evolution of teachers' MTSK during the TPE program. It will assess the enhancement of MTSK, providing insights into progress, key findings, and teacher feedback across each program phase. Finally, the conclusion will address the study's limitations and propose avenues for future research, thereby offering a comprehensive overview of the study's findings and implications.

METHODS

This study employs a qualitative research methodology, specifically adopting a case study approach. Qualitative research focuses on understanding the quality of relationships, activities, situations, or materials (Fraenkel et al., 2012), operating within the interpretive paradigm (Creswell, 2009). While the case study approach does not aim to generalize findings, it offers valuable insights into the intricacies of the problem under investigation (Lune & Berg, 2017). This study uses case studies to delve into teachers' MTSK throughout the TPE program.

Our case study accentuates the pivotal role of mentoring within PLCs in nurturing MTSK among teachers during the planning and execution of teaching practices. Specifically, we focus on the initial development of MTSK and its subsequent evolution through PLCs activities. MTSK is elucidated through narratives, as suggested by Masina and Mbokazi (2023) and Picado-Alfar et al. (2022). These narratives represent a reflective process, capturing changes in the teaching and learning journey over time. It is anticipated that teachers' MTSK narratives will offer valuable insights into their capacity for self-reflection on their teaching practices (Picado-Alfar et al., 2022).

The study design schema is illustrated in Figure 1, providing a visual representation of the research methodology and its various components.

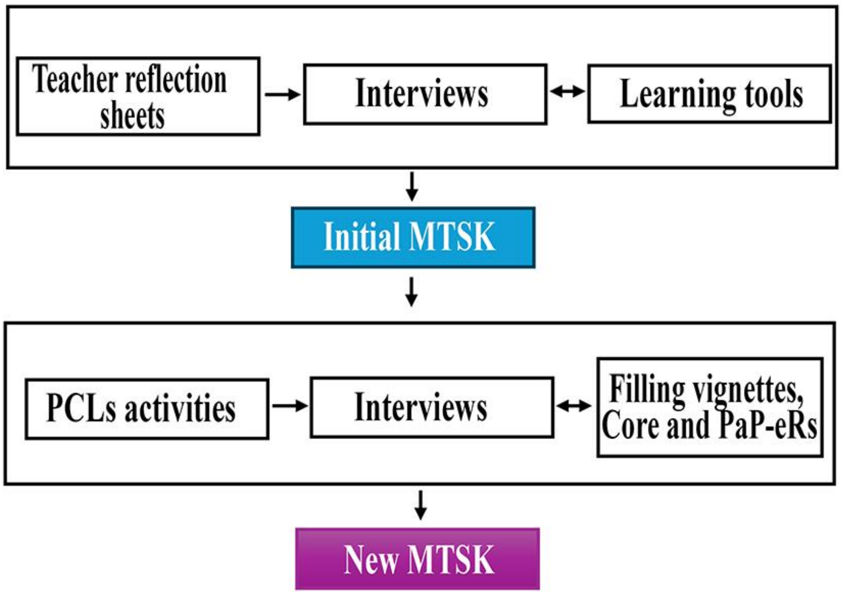


Figure 1. The Study Design

Figure 1 depicts the data collection process utilized in this study, which comprises teacher reflection sheets, learning tools, and interviews. These data were subjected to content analysis, a method capable of uncovering latent aspects of communication present in written text (White & Marsh, 2006). Specifically, the initials MTSK were derived from the content analysis of learning reflections and the preparation of learning tools, focusing on the topic of permutation and combination.

Moreover, synchronous action activities were conducted via video conference with the teaching lecturer, facilitating intervention through interactive learning activities within PLCs. PLCs serve as platforms where teachers can collaboratively discuss, share, and reflect on their teaching experiences in mathematics. Through PLCs, educators have the opportunity to enhance their understanding of Pedagogical Content Knowledge (PCK) in mathematics teaching (Little, 2020) and engage in critical self-reflection on their instructional practices (Chauraya & Brodie, 2017). It is pertinent to note that all activities were conducted online, and access to the LMS was unavailable. Therefore, coordination with the IT administrator of TPE was necessary to procure data on teachers' overall activities on the LMS. Interviews with the three teachers were conducted online at different times, and their responses were analyzed and compared with their activities during video conference recordings.

Participants

The research subjects consisted of mathematics teachers participating in the TPE in-service program organized by the Study Programme of Pendidikan Profesi Guru Universitas Pattimura. The duration of the program was six months, conducted entirely online. Participants were selected based on surveys assessing their Content Knowledge (CK) and Pedagogical Content Knowledge (PCK) relevant to teaching all courses in the TPE curriculum structure, as well as their teaching experience and willingness to participate in the study.

Following the survey results and activities on the LMS platform, participants' mathematics knowledge and pedagogic knowledge were evaluated and classified into three groups: 11 with low ratings, 11 with medium ratings, and 11 with high ratings. This stratification aimed to account for variations in the understanding levels of the selected teachers. Given the nature of the case study research, three teachers were observed, selected from a pool of 33 teachers spanning 13 provinces in Indonesia. All three teachers consented to participate in the study, originating from different suburban schools. The topic of permutations and combinations was chosen as one of the focal points of their teaching practice activities.

Two of the teachers had teaching experience at the Vocational High School level, while the remaining teacher taught at the Senior High School level. Their involvement in mathematics teachers' self-development activities was deemed essential, as indicated by their initial data upon enrollment in the TPE program. Table 1 presents the demographics of the participants.

Table 1. Demographic Information of the Participants

Subject	Gender	Levels are based on Summative Tests	School Status	Ages (years)	Province	Background of Education	Teaching Experience (years)
IST1	Female	High (rank 2)	Private	36	East Java	Bachelors in Mathematics Education	14
IST2	Female	Intermediate (rank 13)	Private	34	West Java	Bachelors in Mathematics Education	10
IST3	Male	Low (rank 20)	State	39	South Sumatra	Bachelors in Mathematics Education	11

Instrument

The research instruments utilized in this study included semi-structured interview guides complementing the quantitative data derived from teachers' reflections, learning tools, observation sheets, and Vignette, CoRe, and PaP-eRs sheets utilized in teaching permutation and combination topics. Notably, the Vignette, CoRe, and PaP-eRs instruments were adapted from Matitaputty et al. (2022). Five sets of questions were crafted based on the MTSK model framework. To ensure the validity and appropriateness of the semi-structured interview guide, three experts evaluated it. Subsequent revisions were made based on their ratings, comments, and suggestions.

Semi-structured interviews were conducted at each stage of the research process to complement other qualitative findings. Table 2 provides an overview of the instruments employed at each study stage.

Data Collection

Data collection occurred at all four research stages, encompassing the initial problem identification stage, the initial MTSK identification stage, the MTSK change stage, and the MTSK consistency evaluation stage. Table 2 delineates the data collection process across these stages, providing a comprehensive overview of the research trajectory and the corresponding data collection methods utilized at each juncture.

Table 2. Data Collection Process

Stage	Description	Instrument	Objective	Study data	Results
First	Identifying learning obstacle in teaching and learning permutations and combinations	Reflection sheet in teaching mathematics on LMS Interview guidelines.	Pay attention to the problems faced in teaching mathematics	Results of filling in LMS teacher reflection Interview results.	Teachers experience problems in teaching combinatorics topics
Second	Identification of initial MTSK	Learning tools (Lesson plan, students' worksheet, textbook and assessment) Interview guidelines.	Reading initial MTSK in teaching mathematics	Learning tools analysis results Interview result	Initial MTSK of mathematics teachers
Third	Analyzing the implementation on MTSK in teaching permutation and combination	Observation sheet Interview guidelines	Focus on MTSK in learning practices	Observations results of learning implementation Interview result	Teachers are able to teach permutations and combinations Teachers have new MTSK
Fourth	Require teachers to fill in Vignette, CoRe, and PaP-eRs to ensure knowledge in teaching permutation and combination	Vignette, Core and PaP-eRs Interview guidelines	Read the changes and consistency of MTSK in teaching permutation and combination	Results of filling in the vignette, CoRe and PaP-eRs MTSK interview results of teachers	Teachers have MTSK in teaching permutations and combinations Search for consistency in new MTSK

Data Analysis

In each stage of the study, four types of information were analyzed. Firstly, qualitative data from teacher reflection sheets and interviews were scrutinized to identify themes by categorizing the content of reflections about learning obstacles encountered in teaching permutations and combinations. These obstacles were classified based on findings from Jatmiko et al. (2017), Nopriana et al. (2023), and Sukoriyanto et al. (2016) for student learning obstacles, and from Muhazir and Retnawati (2020) and Nurlaily et al. (2019) for teachers' learning obstacles. Three types of learning obstacles were delineated through their respective codes: General Learning Obstacles (GLO), Specific Learning Obstacles (SLO), and Evaluation Learning Obstacles (ELO). GLO refers to general impediments in learning permutation and combination concepts, SLO pertains to explicit descriptions of students' learning obstacles, and ELO involves the identification of students' learning obstacles and evaluating areas for improvement.

Secondly, qualitative data was derived from the design of learning tools before teaching practice. Thirdly, data obtained from completing Vignettes, CoRe, and PaP-eRs were analyzed. The third and fourth types of information were supplemented by semi-structured interviews and analyzed using MTSK indicators proposed by Carrillo-Yañez et al. (2018). In the MTSK model, the Mathematical Knowledge (MK) domain is divided into three components: Knowledge of the Topic (KoT), Knowledge of the Structure of Mathematics (KSM), and Knowledge of Practices in Mathematics (KPM). KPM was not identified in any activities or interviews with the subjects. The indicators used for data analysis of the MK domain, specifically KoT and KSM, are detailed in Table 3.

Table 3. Mathematical Knowledge Indicator

MK Components	Indicator	Scale	Evident when the teacher …
Knowledge of Topic (KoT)	KoT1: Knowledge regarding the definitions of permutation and combination	1	Knowing permutation and combination formulas without understanding the basics.
		2	Defines permutation as an arrangement or selection of elements that considers the order. Combination is defined as the arrangement or selection of elements without regard to order. Furthermore, the ability to calculate permutations and combinations in real situations.
		3	Understand the definition of permutation and combination. Ability to differentiate between the concepts of permutation for different and similar elements, as well as cyclic permutations. Ability to apply permutation and combination in the context of discrete mathematics, statistics, probability theory, and various other scientific disciplines.
	KoT2: Knowledge of connecting various representations to solve permutation and combination problems	1	Understand the relationship between various representations in limited permutation and combination problems
		2	Understand the relationship of various representations better
		3	Understand the relationship of various representations in permutation and combination problems flexibly, creatively, and effectively in solving problems.
Knowledge of the Structure of Mathematics (KSM)	KSM1: Knowledge to assess how content is taught and linked to broader areas of knowledge	1	Have a limited understanding of how content is taught and connected to broader disciplines
		2	Have a better understanding of how content is taught and connected to broader areas of knowledge
		3	Have a deep understanding of how content is taught, its relevance to the real world, and the role of content in the larger body of science
	KSM2: Knowledge to explain the different concepts and formulas for	1	Have a very limited basic understanding of the difference between permutation and combination.
		2	Have a better understanding of the difference between permutation and combination.

MK Components	Indicator	Scale	Evident when the teacher …
	permutation and combination	3	Have a deep understanding of the differences between permutation and combination, as well as the ability to apply these concepts in a variety of situations

Carrillo-Yañez et al. (2018) explain that PCK, the second domain of the MTSK model, is intricately connected to mathematics, embodying a specialized form of pedagogical knowledge derived from the realm of mathematics education. PCK consists of three primary components, delineated as Knowledge of Mathematics Teaching (KMT), Knowledge of Features of Learning Mathematics (KFLM), and Knowledge of Mathematics Learning Standards (KMLS). Table 4 presents an overview of these three components comprising the PCK domain within the MTSK model.

Table 4. Pedagogical Content Knowledge Indicators

PCK Components	Indicator	Scale	Evident when the teacher …
Knowledge of Mathematics Teaching (KMT)	KMT1: Knowledge of how to teach permutation and combination (strategies, techniques, tasks, and examples)	1	Uses a limited approach, provides less real-life application of permutations and combinations, and does not provide challenging tasks for students.
		2	Combines a variety of teaching methods, provides examples of real-world applications, and provides assignments that require the application of permutations and combinations in more complex situations.
		3	Using technology, simulations, and games to explain the concept of permutation and combination interactively, discussing with students to build an understanding of the concept, and promoting students to become independent learners.
	KMT2: Knowledge of the use of teaching resources	1	Limited in using conventionally available resources and less adapted to the diverse needs of students.
		2	Consider different types of teaching resources to meet learning needs.
		3	Develop innovative teaching strategies, use a variety of resources, and deeply engage in learning experiences.
Knowledge of Features of Learning Mathematics (KFLM)	KFLM1: Knowledge regarding how to understand the difficulties and misconceptions that students often encounter in solving permutation and combination	1	Not realizing the difficulties students face in understanding the concepts of permutation and combination or the sources of common misconceptions.
		2	Ability to identify difficulties and misconceptions that often arise when students study permutation and combination.
		3	Ability to use various teaching strategies specifically designed to address student difficulties and misconceptions regarding basic concepts and principles of permutation and combination.
	KFLM2: Knowledge to predict interesting things and motivate students to	1	Have an understanding that may be very limited in motivating students to comprehend permutation and combination material.

PCK Components	Indicator	Scale	Evident when the teacher …
	understand permutation and combination materials.	2	Have an understanding of how to predict things that will interest and motivate students to be more structured.
		3	Having an understanding of how to predict interesting things and motivate students to understand permutation and combination material helps create an effective learning environment and facilitate student interest, making learning more effective and meaningful.
Knowledge of Mathematics Learning Standard (KMLS)	KMLS1: Knowledge of content and learning objectives in accordance with curriculum.	1	Have a limited understanding of the content and learning objectives in accordance with the curriculum.
		2	Have an understanding of the content and learning objectives in accordance with the curriculum helps students to be more structured.
		3	Have an understanding of the content and learning objectives according to the curriculum is very effective in creating a strong learning experience.
	KMLS2: Knowledge of the order in which content is presented, reminding students of previous as well as subsequent materials.	1	Lack of understanding of the importance of the order in which material is presented and teaching topics randomly.
		2	Have a basic understanding of how to properly sequence the presentation of material can improve student understanding.
		3	Understand the importance of good order in planning and presenting material according to a logical sequence and ensure continuity in student understanding.

The graphical depiction of MTSK illustrates the evolution of teachers' knowledge over time. This visual representation delineates the trajectory of knowledge advancement, anticipated to trend positively. The methodology employed to craft this graphical portrayal of MTSK is delineated in Figure 2.

The MTSK graphs were created using a structured sequence of steps: (1) Identification of both MK and PCK during immersive learning activities such as vignette completion, creation of CoRe, and PaP-eRs; (2) Categorization and fine-tuning of teacher knowledge according to predefined indicators; (3) Determination of the scale to represent the teachers' perspectives; and (4) Conversion of the identified knowledge into a visual format. Each teacher's graph was individually tailored to highlight their unique knowledge profile, enabling a precise observation of knowledge fluctuations over time.

Figure 2 shows the graphical representation of the MTSK, which consists of the components to build the MTSK, the MTSK chart, and the indicator scale of the MTSK. Each component has two indicators. Each indicator has three scales depicted as circles. These circles signify the extent of the scale for each element, with circle 1 denoting a scale of 1 (1), circle 2 representing a scale of 2 (2), and circle 3 indicating a scale of 3 (3). The color scheme distinguishes between teachers' MTSK in the second stage (red) and MTSK in the third stage (blue).

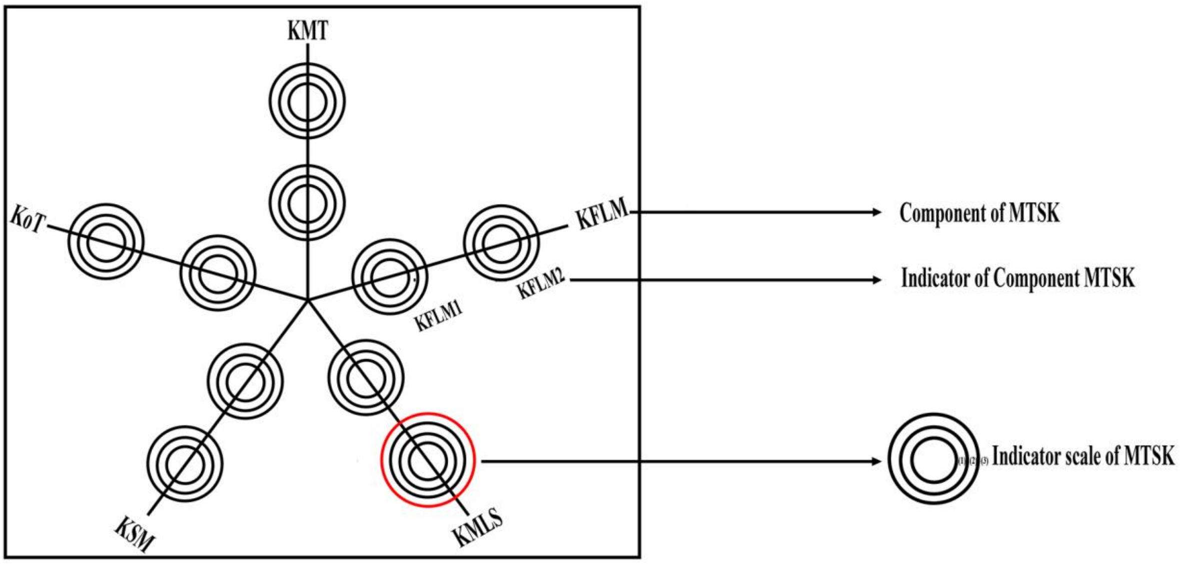


Figure 2. MTSK Graphical Representation

RESULTS AND DISCUSSION

Aligned with our research objectives, we anticipate that teachers will acquire fresh insights through participation in the TPE Programme, particularly in the domain of teaching permutation and combination topics. Our primary aim is to elucidate teachers' knowledge within the framework of the MTSK model as they engage in the entirety of the TPE program, focusing on the instruction of permutations and combinations. This section is structured into four stages: First, identifying the hurdles encountered in teaching and learning permutations and combinations; second, unveiling the initial MTSK derived from preparing instructional materials for teaching permutations and combinations; third, conducting analysis and presentation of MTSK evolution in teaching practices, delineating shifts in teachers' MTSK; and lastly, presenting the assessment outcomes of MTSK and evaluating the coherence of the newly acquired knowledge structures.

First Stage

At this stage, we conducted an analysis of data gathered from teachers' reflection sheets and interviews, aiming to elucidate the obstacles faced by both teachers and students in comprehending and instructing the intricacies of permutation and combination topics. Through material-deepening activities, teachers were tasked with providing reflections grounded in their teaching experiences and comprehension of combinatorics and statistics learning modules. Notably, our findings unveiled discrepancies in the knowledge base of three distinct teachers regarding identifying obstacles encountered in both teaching and learning permutations and combinations, as illustrated in Table 5. The percentages presented therein are calculated by dividing the number of instances of each identified learning obstacle by the total number of obstacles documented in the teachers' reflection sheets.

Table 5. Proportions of the Learning Obstacle

Learning Obstacle	IST1	IST2	IST3
GLO	1 (5%)	3 (16%)	2 (12%)
SLO	16 (84%)	14 (78%)	15 (88%)
ELO	2 (11(%)	1 (6%)	-
Total	**19 (100%)**	**18 (100%)**	**17(100%)**

Table 5 illustrates that each of the three teachers is proficient in documenting particular obstacles encountered by students and teachers in teaching permutations and combinations (SLO).

Independently Worksheet 1

Topic of textbook		Textbook 4: Combinatoric and Statistic
Learning activity		1. Counting rules, permutations, and combinations 2. Probability theory 3. Central tendency 4. Learning Combinatoric and statistics
No.	**Point of reflections**	**Response**
2	List of material that is difficult to understand in this module	1. Students confuse additional rules and multiplication rules. 2. Students difficult to understand definition of permutation is the number of ways to create an arrangement with a certain number of elements from the available elements by paying attention to the order. 3. Students solve combinations problems using permutation formula.
3	List of the learning resources	1. Using the Ministry of Education textbook for class XII in 2018 2. Module of Mathematics High School for grade 12

Figure 3. IST3 Reflection Sheet

However, it is noteworthy that IST1, while identifying specific student learning obstacles, needed help articulating the feedback necessary to enhance student comprehension. Furthermore, IST3 has yet to broaden his understanding by leveraging diverse learning resources. A partial excerpt from his reflection sheet is depicted in Figure 3.

In contrast to IST3, IST1 demonstrated the ability to elucidate student learning obstacles effectively and evaluate their teaching strategies. They exhibited proficiency in assessing the efficacy of their teaching interventions, discerning what was effective, and identifying areas for improvement. This capability is evident from IST1's responses during the interview process.

> *“Students confuse between permutation and combination. Especially in combination with repetition, they cannot understand the concept. They make calculation errors and sometimes have difficulty composing mathematical sentences. I think they must do various exercises, ranging from low to high difficulty levels. I can provide interactive activities like games or hands-on activities".*

Furthermore, IST2 shared her experience, revealing that her learning materials adhered strictly to the conventional lesson plan format and did not incorporate technology. She also acknowledged dominating the learning process by providing explanations herself, driven by time constraints. IST2's account underscores her comprehension of evaluating learning obstacles. Based on these findings, initial assumptions were made regarding the teachers' initial MTSK. It was hypothesized that the teachers' MTSK might need to be developed due to various challenges associated with obstacles encountered in learning permutations and combinations. This experiential evidence highlights limitations in the teachers' knowledge concerning the effective instruction of permutation and combination topics.

Second Stage

The findings from the initial stage concerning problem identification were leveraged to delineate the MTSK about the permutation and combination topic. An overview of MTSK was obtained by analyzing lesson plans, teaching materials, student worksheets, media, and assessments, guided by semi-structured interviews employing MTSK indicators. Table 6 summarizes this MTSK framework, elucidating the teachers' comprehension and proficiency levels regarding the instruction of permutation and combination concepts.

Table 6. MTSK Identification of Teachers in the Second Stage

Subject	MTSK Component	MTSK Interview Results of Teachers	Scale
IST1	KoT1	In my opinion, permutation is the arrangement of objects in a set considering order, while combination is the arrangement of objects in a set without considering order.	2
	KoT2	I prefer to use the permutation and combination questions in the book which are equipped with interesting pictures.	2
	KSM1	I explain to students that the concepts of permutation and combination are related to the concept of chance.	1
	KSM2	I explain to students that permutation rules consider order while combination rules do not.	1
	KMT1	After explaining the concept, I classify students into heterogeneous groups to solve problems.	1
	KMT2	I use textbooks at school to teach permutation and combination.	1
	KFLM1	Students often find it difficult to solve permutation and combination problems and often interchange formula.	2
	KFLM2	Some of my students do not really like learning permutation and combination due to several calculations.	1
	KMLS1	The syllabus usually starts with the concept of permutation and forms of permutation, followed by content.	1
	KMLS2	Permutation and combination material is part of the rules of enumeration. After explaining the concept of permutation, I continue with combination. I usually review the place-filling rules and factorial notation before explaining both topics.	2
IST2	KoT1	I explained that permutation pays attention to order while combination does not. I usually relate the topics to everyday life problems.	2
	KoT2	I guide students to write down what they know and ask about the problem. I subsequently direct them to use a specific formula.	1
	KSM1	The concepts of permutation and combination can be used to solve problems related to probability.	1
	KSM2	I explain to students that permutation pays attention to order while combination does not.	1
	KMT1	I usually present the material and give practice questions.	1
	KMT2	I used textbooks provided by the school and some references that I could obtain for free.	1
	KFLM1	Students find it difficult to solve permutation and combination problems and often interchange formula.	2

Subject	MTSK Component	MTSK Interview Results of Teachers	Scale
	KFLM2	I know that my students can understand when I explain the concept first and give examples.	1
	KMLS1	Permutation and combination material is given before opportunity material.	1
	KMLS2	After students understand the rules for filling in places and factorial notation, the material on permutation and combination is explained.	1
	KoT1	I teach students that the permutation formula for r elements from n elements is $P(n,r) = \frac{n!}{(n-r)!}$ and the formula for the combination of n elements is $C(n,r) = \frac{n!}{r!(n-r)!}$.	1
	KoT2	After presenting the formula, I ask students to apply by solving related problems.	1
	KSM1	The concepts of permutation and combination are related to the definition of factorial.	1
	KSM2	I teach these two topics separately, I first explain the concept and forms of permutation followed by the concept of combination.	1
IST3	KMT1	I teach permutation and combination starting by explaining basic concepts including factorials and definitions, and giving students practice questions to work on in groups.	1
	KMT2	I use the 2013 curriculum textbook.	1
	KFLM1	Students do not understand the definition of factorial of a positive integer n, and detect errors when dividing using factorial. Example: $\frac{10!}{5!} = 2!$	1
	KFLM2	I explain better and give examples for students to understand the material on permutation and combination.	1
	KMLS1	Combination material is taught after students understand permutation. Therefore, the two materials are taught separately.	1
	KMLS2	Permutation material is taught after combination material	1

The outcomes were depicted through a graphical format, utilizing indicators gleaned from interviews consistent with the predefined indicators established in the theoretical study. Figure 4 showcases a graphical representation of MTSK, illustrating the teachers' knowledge structure concerning permutation and combination topics.

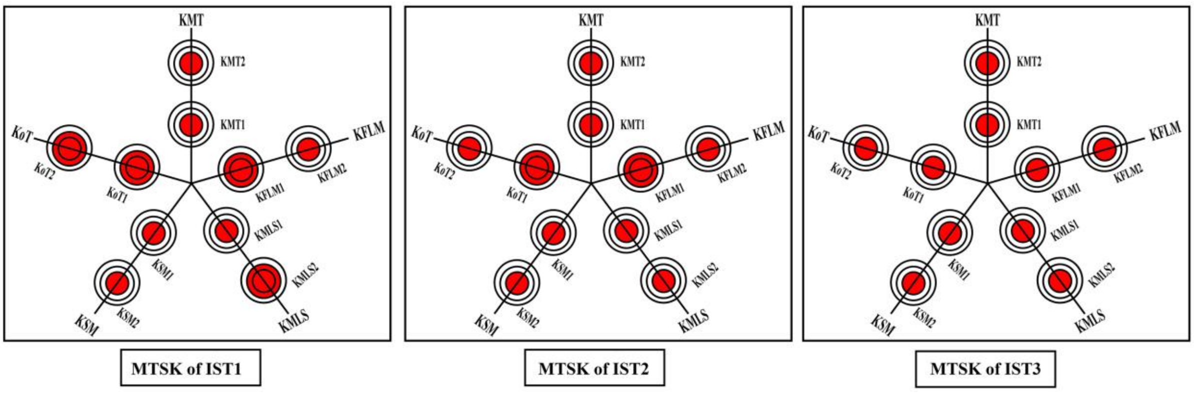


Figure 4. MTSK of Teachers when studying independently

Figure 4 indicates that certain aspects of MTSK continue to align with the teachers' reflections. Notable developments in teacher MTSK were observed, particularly in the case of IST1, who exhibited an understanding extending beyond mere formulas to grasp the conceptual framework of organizing elements within a set (KoT1). Indeed, it is noteworthy that she effectively employed figure representations to enhance students' visual understanding (KoT2) comprehensively.

Furthermore, IST1 demonstrated an adept understanding of students' misconceptions (KFLM1), addressing them by employing limited repetition of fundamental concepts before delving deeper into permutation and combination concepts (KMLS2). Conversely, IST2 exhibited a broader comprehension of permutations and combinations, emphasizing the distinction between them and their application in real-world scenarios (KoT1). IST2 also adeptly identified students' difficulties and grasped familiar sources of misconceptions, particularly regarding the differentiation between permutations and combinations (KMLS2).

In contrast, IST3 showed limited progress across all MTSK components. His grasp of permutations and combinations remained rudimentary, relying primarily on simplistic formulas without delving into underlying concepts (KoT1). Furthermore, his understanding of representation techniques remained constrained, needing more exploration beyond formulaic approaches (KoT2). Moreover, IST3's comprehension of students' difficulties and misconceptions in permutation and combination topics remained limited, primarily centered on errors related to factorial calculations (KFLM1). Additionally, IST3 displayed a limited understanding of the sequencing of content presentation, failing to emphasize previous material or utilize repetition techniques to reinforce prior concepts (KMLS1).

Third Stage

During this stage, a series of targeted interventions were implemented, allowing the subjects to discuss the preparation of learning tools with supervisors, colleagues, and tutors. Subsequently, the learning materials underwent scrutiny and approval by the supervisor and tutor. Furthermore, the subjects' teaching activities were observed, and feedback was provided to facilitate learning improvement.

This phase witnessed notable development in MTSK as teaching methodologies transitioned towards a more interactive, activity-centered approach to engage students in the learning process actively. Table 7 provides an overview of MTSK at this third stage, highlighting the advancements made in teachers' knowledge structures.

Table 7. MTSK Identification of Teachers in the Third Stage

Subject	MTSK Component	MTSK Interview Results of Teachers	Scale
IST1	KoT1	My mentor provided input on defining permutation and combination. I had to modify this definition by giving examples of daily life problems that students could encounter, such as issues related to the election of OSIS administrators and the formation of the management.	2
	KoT2	One of the fellow teachers suggested presenting permutation and combination problems using visual aids showing activities of arranging and selection of cards (without replacement). This method includes presenting the results in the form of a table or arrow diagram to guide students in comprehending permutation and combination formulas and recognizing the difference between these two concepts.	3

Subject	MTSK Component	MTSK Interview Results of Teachers	Scale
	KSM1	During the presentation of this material in small groups, I received questions about the relevance of permutation and combination. I responded to students that these concepts were crucial in statistics for calculating various ways of selecting samples from the population and for computing probabilities. I also explained how the relationship between the concept of combination and Newton Binomial lied in the use of combination to calculate the coefficients of terms in the binomial expansion.	3
	KSM2	I explained the different concepts of permutation and combination by using the Arrange activity and picking up some playing cards. I subsequently asked students to analyze the differences between these activities of arranging and drawing cards and determine whether there was any permutation or combination problem.	3
	KMT1	I initiated the learning process with a brainstorming game and subsequently delivered the material by using digital teaching aids. I organized students in groups to complete worksheets, and guided in analyzing the differences between permutation and combination. My mentor provided valuable advice to evaluate students and give feedback during learning process.	3
	KMT2	My mentor suggested strategies to enhance student engagement in learning. I was also given input on the syntax of PBL model, and promoted to incorporate the characteristics of 21st-century learning including critical thinking, collaboration, communication, and creativity, into the learning design that I created. During teaching practice activities, I developed teaching materials in the form of flip books, grouped students to complete worksheets, and directed them to analyze the differences between permutation and combination. I also evaluated and provided feedback on work presentations. Further discussions were facilitated through Google Classroom.	3
	KFLM1	I realized misunderstanding of students when asked to calculate the possibility of arranging 2 aces from the 4 aces available, and responded with 6 possible ways. To address this, I used the activity of selecting and arranging playing cards.	3
	KFLM2	Based on input from the mentor, I tried to challenge students by introducing a quiz featuring several permutation and combination problems at the onset of the lesson.	2
	KMLS1	From the start, I discussed this topic with my mentor and colleagues who advised me to present these two topics in an integrated manner. This helped me to achieve the learning objectives. Furthermore, through the activity of solving contextual problems, students were able to understand the differences between permutation and combination. I carried out learning diagnostic, problem-solving skills, and attitude assessments, while ensuring the students worked both collectively and individually throughout the lesson period.	3
	KMLS2	My mentor reminded me to pay attention to the order of the material and teach sequentially. Even though I explained these two concepts in an integrated manner, I ensured there was a logical flow between concepts or sub-topics. For instance, I introduced permutation forms accompanied by definitions and examples, and subsequently conveyed that the material under study was related to probability theory, modeling, and analysis.	3

Subject	MTSK Component	MTSK Interview Results of Teachers	Scale
	KoT1	My mentor elaborated on the importance of careful sequencing and basic concepts that students needed to understand. During teaching practice, I explained the rules for enumeration, which included rules for filling positions, addition, multiplication, permutation of different elements, permutation of multiple identical elements, cyclic permutation, and combination. Permutation entails arranging elements in a certain order while combination includes selecting elements without regard to order. I showed this difference by using discussions about forming vocal teams and recording cellphone numbers.	2
	KoT2	One of my colleagues suggested I should incorporate various representations that suit student interests in presenting the concepts of permutation and combination. Consequently, I selected pictorial, verbal, and symbolic representations to clarify these concepts.	2
	KSM1	During my experience teaching small classes, I received suggestions to pay attention to the relevance of permutation and combination in solving problems related to understanding phenomena and modeling situations in various scientific contexts. After mentioning the relationship between these concepts and opportunities, my mentor added that the concepts could also be applied to model consumer choices. This would aid in understanding the various ways of selecting products and calculating the number of possible purchase combination.	3
IST2	KSM2	One of my colleagues suggested that I clarify the differences between permutation and combination using simple problems, enabling students to differentiate between situations where order matters and those where it does not.	2
	KMT1	With guidance from my mentor, my learning design focused more on building student understanding through simple problems and subsequent in-depth discussions. This approach, coupled with constructive feedback, helped students understand the concept of permutation and combination.	2
	KMT2	I received important input on using the model from mentors and peers. This made me understand the use of problem-based learning models and scientific learning approaches. I facilitated classical discussions and provided teaching materials, and students worked in groups to address questions, followed by presentation in front of the class.	2
	KFLM1	During teaching practice, I observed common mistakes made by students when mentioning permutations of n elements from r elements. I also noticed student errors in writing the permutation notation as $_3P_5$ and mentioning $_5P_2$ as "P five two". I prefer to use simple problems related to student daily lives to effectively convey the concept of permutation and combination.	3
	KFLM2	My mentor always elaborated the importance of appreciating student answers. During classical discussions, students were promoted to express appreciation when one of peers respond to questions.	2
	KMLS1	My mentor suggested I adapt learning objectives to personal abilities. I recognized that while expecting students to analyze, differentiate, and solve permutation and combination problems, it was crucial to present the content in a structured manner.	2

Subject	MTSK Component	MTSK Interview Results of Teachers	Scale
	KMLS2	One of my colleagues offered a helpful suggestion to introduce enumeration rules and factorial definitions before delving into permutation. From the beginning of the teaching practice, I reminded students that the factorial concept would be used in solving permutation and combination problems. When explaining concepts, I engaged in reflective discussions to assess student understanding. My mentor also advised me to ensure the continuity of student comprehension of subsequent concepts, particularly in statistics and probability theory.	3
	KoT1	I realized that the factorial concept serves as foundational basis for understanding the definition of permutation and combination. My mentor promoted me to provide a solid understanding of this concept. Consequently, I defined permutation as the arrangement of elements with consideration to order, and combination, the arrangement of elements without considering order. I used the example of composing and selecting members of a competition team to explain the difference between these two definitions.	2
	KoT2	My mentor fully supported my approach of explaining permutation and combination using daily life problems and experiences of students.	2
	KSM1	I explained to students that permutation and combination could be used to solve daily problems such as calculating the number of vehicle codes in one area.	2
	KSM2	I received feedback to introduce the formulas for permutation and combination, starting with easy questions and gradually progressing to more challenging ones.	3
	KMT1	My mentor advised me to adopt a problem-based learning model by first presenting daily life problems and gradually developing the concepts of permutation and combination. I also created digital teaching materials to promote independent learning of students.	2
IST3	KMT2	My mentor suggested I divided the material on permutation and combination into two sessions to ensure student understanding. During teaching practice, I provided learning resources using book creators for both topics. This approach simplifies the learning process for students and includes fundamental concepts with real-life examples. Group worksheets were subsequently assigned to students to present work and receive feedback.	3
	KFLM1	My mentor advised I consistently define factorial at the beginning of delivering permutation and combination material. In this case, I explained the calculation procedure for factorials to eliminate student errors. I also observed student errors in applying permutation formula to solve combination problems. I guided students by consistently explaining both concepts.	2
	KFLM2	I was promoted to present questions related to applying permutation and combination in the book creator in order to promote student active participation in the learning process.	2
	KMLS1	My mentor advised me to explain these two materials separately in order to achieve learning objectives, namely student understanding and the use of permutation in solving problems of the same element, different elements, and cyclical permutation. Students also acquired concepts of combination to solve related problems.	1

Subject	MTSK Component	MTSK Interview Results of Teachers	Scale
	KMLS2	My mentor promoted me to pay close attention to the basic concepts of these two topics. I explained concepts of permutation for different elements, same element, and cyclical permutation. When covering the topic of combination, I reminded students of these two concepts and the differences.	2

Figure 5 shows that it is evident that specific components of the teachers' MTSK have begun to evolve. The PLCs group observed notable developments in the teachers' MTSK during their participation. Specifically, IST1 demonstrated an enhanced understanding of various aspects, such as relationship between different representations in permutation and combination problems (KoT2), the relevance of content to real-world applications across different scientific fields (KSM1), ability to articulate the fundamental distinctions between permutations and combinations (KSM2), utilization of interactive approaches, such as providing feedback to underscore conceptual understanding (KMT1), integration of technology to augment learning experiences, including tools like flipbooks, Google Classroom, and interactive Quiz games (KMT2, KFLM2), identification and remediation of students' specific misconceptions (KFLM1), implementation of continuous evaluation strategies (KMLS1), and organization of material presentation in an exemplary sequence (KFLM2). These advancements underscore the progressive development of MTSK among the teachers, particularly in integrating diverse instructional strategies, leveraging technology, and effectively addressing student misconceptions.

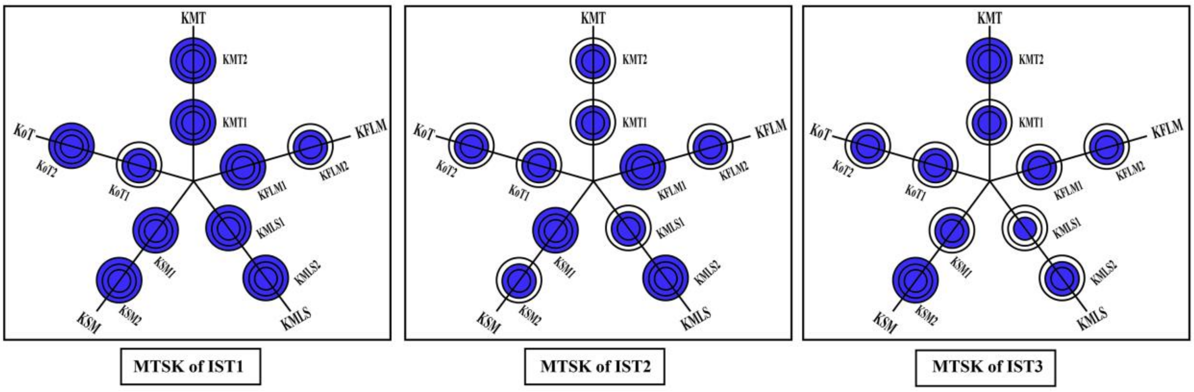


Figure 5 MTSK when Studying with a PLC group.

IST2 has demonstrated a well-developed understanding across various dimensions, as evidenced by grasping the relationship between different representations in permutation and combination problems (KoT2), integrating knowledge across diverse scientific fields and its application to real-world scenarios (KSM1), discerning the distinction between permutation and combination concepts (KSM2), employing a combination of learning methods to enhance instruction (KMT1), selecting and effectively integrating appropriate teaching resources into instructional practices (KMT2), identifying the sources of student misconceptions (KFLM1), employing diverse teaching strategies to motivate students (KFLM2), ensuring the alignment of instructional material with the curriculum (KMLS1), and utilizing repetition techniques effectively to reinforce learning (KMLS2). These competencies illustrate IST2's comprehensive understanding and adeptness in integrating diverse instructional strategies, addressing student misconceptions, and ensuring alignment with curriculum objectives.

Furthermore, IST3 has demonstrated an enhanced understanding across all MTSK components, albeit at a moderate level. Specifically, IST3 demonstrates a broader understanding of concepts by contextualizing them within real-world situations (KoT1), displays some limitations in adapting different representations (KoT2), identifies the usefulness and relevance of content to real-world applications (KSM1), articulates the fundamental differences between permutations and combinations and their application in real-world scenarios (KSM2), integrates various teaching methods effectively (KMT1), utilizes a variety of teaching resources (KMT2), identifies the sources of misconceptions and conducts corrective evaluations (KFLM1), makes efforts to engage students by presenting material in an interesting manner (KFLM2), and demonstrates a good understanding of the sequencing of material presentation (KMLS2). These developments signify IST3's progress in comprehending and implementing various teaching strategies and addressing student misconceptions, thereby enhancing the overall effectiveness of instructional practices.

Fourth Stage

In this phase, a consistency checks on MTSK was conducted. Given that the three teachers had already attained a robust understanding of teaching permutation and combination material during their participation in the TPE courses, participants willingly completed Vignette, Co-Re, and PaP-eRs sheets to offer comprehensive insights into their teaching knowledge. The findings indicated that the MTSK of the teachers remained consistent with what was observed in the third stage. Figure 6 illustrates a graphical representation of PCK consistency for teachers in stage four.

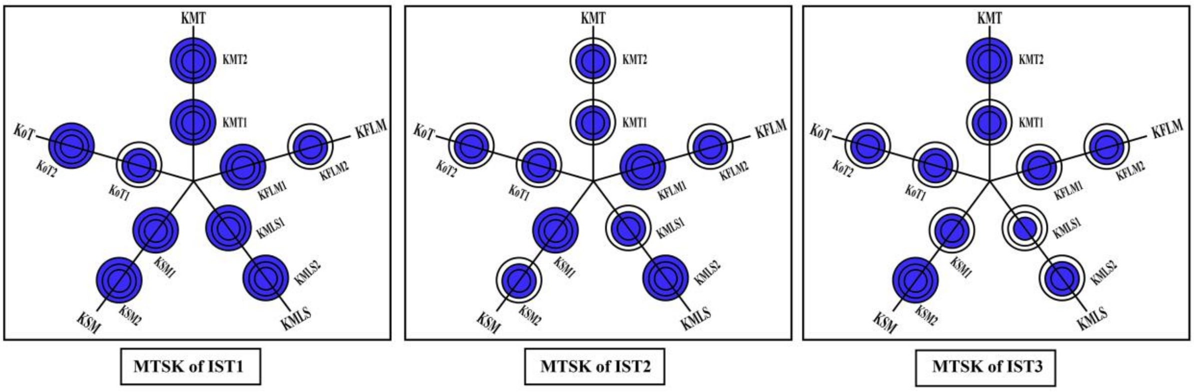


Figure 6. New MTSK Consistency

Figure 6 shows the three teachers' consistent progression of MTSK development, trending towards a more advanced understanding. IST1 exhibits a high level of development in numerous MTSK components, although KoT1 and KFLM2 remain at a moderate level. Conversely, IST2 demonstrates the development of several MTSK components at both moderate and high levels, with only KoT1 yet to see substantial development. In the case of IST3, all MTSK components have evolved, albeit at a moderate level, with KSM2 and KMT2 being particularly well-developed.

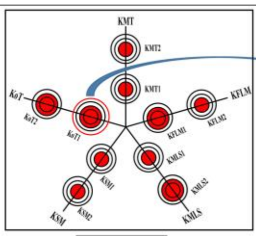
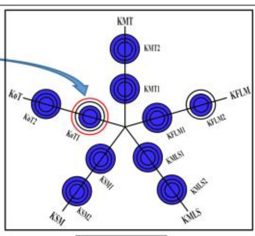
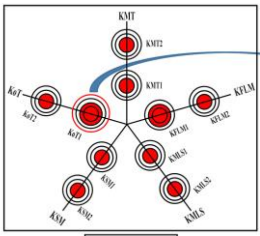
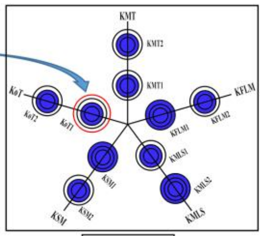
Changes in MTSK of Teachers

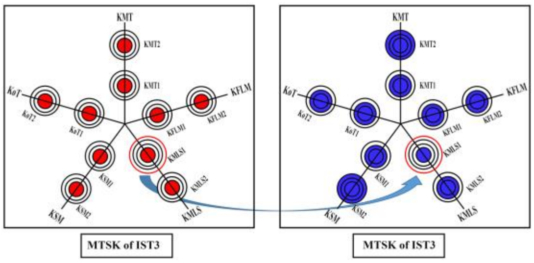
The transformation of MTSK from its initial state was notably evident as teachers engaged in TPE activities during the first and second stages. Collaborative interactions within the learning community, involving supervisors, colleagues, and tutors, facilitated the preparation of teaching materials during the third stage. Technology integration into teaching methodologies further enhanced the learning experience, fostering greater student engagement. Additionally, content knowledge expanded by incorporating real-life examples related to permutation and combination material, enriching the learning environment and deepening students' understanding.

The consistency of MTSK observed at the third stage was carried forward into the fourth stage through various activities, including PLCs engagements and responses to student error cases in problem-solving (Vignette), articulation of knowledge in content representation (CoRe), and detailed accounts of teaching experiences related to permutation and combination (PaP-eRs). These activities aimed to elucidate and correlate teachers' implicit knowledge with their teaching practices, offering comprehensive insights into the PCK (Bertram, 2014).

Recognizing the changes in MTSK could prompt schools and educational systems to consider the "other side" of professional development programs, with implications for understanding mathematics teaching. By comprehending MTSK, teachers can effectively tackle challenges in implementing learning, identify needs for participating in relevant personal development activities, and support students with diverse backgrounds and abilities. The graphical representation portrays the transformation in MTSK, symbolizing progress and enhancement. The transformation in MTSK for the three teachers is detailed in Table 8.

Table 8. Initial Teacher MTSK (Independent Learning) becomes New MTSK (PLC Joint Learning)

MTSK Stage 2	MTSK Stage 3	Description
 MTSK of IST1	 MTSK of IST1	The new MTSK IST1 showed the development of knowledge about topics, particularly in understanding the relationships among various representations when solving permutation and combination problems. Knowledge about mathematical structures from IST1 was developed through the ability to assess content with other fields of science and being able to explain the differences in concepts and formulas. IST1 was. able to develop innovative teaching strategies to address student difficulties and misconceptions, thereby creating effective and meaningful learning experiences. Knowledge about the sequence of content and learning objectives was intertwined when the content was presented in an integrated manner. Only knowledge of the definition was underdeveloped (shown by the green circle).
 MTSK of IST2	 MTSK of IST2	The new MTSK IST2 showed improvements in understanding the relationship component of representation when explaining permutation and combination. IST2 had a good understanding of the interconnections of representations and was able to comprehend as well as relate content to the real world and other fields of science. IST2 was able to build student understanding by using various methods supported by teaching resources that suit student needs. IST2 understood student difficulties and misconceptions and helped by creating effective learning. IST2 had not



experienced development regarding the definition of permutation and combination (shown by green circles). The new MTSK showed by IST3 is quite a development in terms of understanding the differences regarding the definitions of permutations and combinations and the application in the context of probability theory. IST3 understands the use of innovative resources and teaching strategies. IST3 understands difficulties and misconceptions of students and responds better. Understanding of the learning objectives and sequence of IST3 material is in accordance with the curriculum but less structured because IST3 works on this material separately (shown by the green circle).

Understanding MTSK could enable educational institutions to recognize the significance of collaborative teaching efforts to share experiences and ideas, ultimately enhancing student learning outcomes. MTSK underwent evolution through the guidance and input received from supervisors, colleagues, and tutors during discussions surrounding the preparation of teaching materials. Throughout the teaching process, teachers had the opportunity to develop their knowledge further by integrating technology into learning experiences, thereby fostering a more engaging and enjoyable educational environment for students. Additionally, content knowledge was enriched by incorporating real-life examples pertinent to permutation and combination topics. On the other hand, MTSK was significantly shaped by the input garnered from discussions with supervisors, colleagues, and tutors, particularly in the context of classroom performance (Carrillo et al., 2019).

In the MK domain, IST1, IST2, and IST3 exhibited similarities in understanding the KoT components. All three teachers defined permutation and combination as the arrangement or selection of elements considering the order within a set. This suggests that their knowledge of the topic, particularly in terms of definitions, needed to be developed as they struggled to apply these concepts in more complex scenarios. This aligns with findings from Lockwood et al. (2020) and Soto et al. (2022), who emphasized the need to enhance teaching approaches for combinatorics problems based on definitions to foster opportunities for deeper and more thought-rich mathematics learning.

A common aspect shared among the three teachers in the PCK domain was the KFLM component. Each teacher demonstrated the ability to anticipate factors motivating students to comprehend permutation and combination concepts. This was exemplified by statements such as: "I challenge students by conducting quizzes on various permutation and combination problems at the beginning of the lesson" (IST1); "I encourage students to participate in class discussions and contribute to discussions on the differences between permutation and combination in simple conversations" (IST2); and "I integrate questions related to the practical application of permutation and combination into the book creator, actively engaging students in the learning process" (IST3). Primarily, PLCs offer a platform for teachers to engage in reflective practices together, intentionally and systematically, fostering collective and sustainable changes in their teaching approaches (Chauraya & Brodie, 2017). The role of PLCs in influencing changes in teachers' teaching practices often hinges on teachers' self-reports of classroom experiences (Vescio et al., 2007). The existence of a new MTSK, reflecting the exchange of knowledge and collaborative reflection on teaching practices within PLC discussions, suggests that teachers can gain fresh insights and refine their pedagogical approaches through collective professional engagement.

The descriptions provided by the teachers notably differed from their initial accounts of teaching permutation and combination concepts. This shift was evident in statements such as: "I utilize the Problem-Based Learning (PBL) model with learning steps tailored to 21st-century learning, emphasizing critical thinking, collaboration, communication, and creativity." These comments on cultivating a new MTSK underscored the acquisition of additional knowledge stemming from the learning models employed by teachers in both instructional design and implementation.

The PLCs played a pivotal role in fostering this transformation, resulting in a new and positive MTSK, as evidenced by the innovative approaches adopted in teaching permutation and combination concepts. Furthermore, teachers demonstrated incredible structure in delivering materials tailored to student needs. On the other hand, PLCs have been shown to promote effective mathematics learning, leading to improved outcomes (Graham, 2007; Sigurdardóttir, 2010). It was anticipated that this positive influence on mathematics teaching practices would, in turn, drive the development of MTSK in a more favorable direction.

MTSK Narrative

The analysis of the four stages illustrates the development of MTSK through the experiences of three teachers: IST1, IST2, and IST3. IST1 emerged as an exemplar in teaching permutation and combination topics, demonstrating expertise through several vital practices. Firstly, IST1 adeptly explained the distinctions between permutation and combination, providing precise definitions and relating them to real-life examples. Furthermore, IST1 effectively integrated these concepts into practical situations, such as arranging and selecting playing cards, to enhance student understanding. Collaboration and problem-solving skills were nurtured by organizing students into small groups, leading to observable improvements in students' abilities to formulate and solve permutation and combination problems. These findings are corroborated by previous research indicating that teachers can effectively teach permutation and combination topics through various activating teaching strategies (Matitaputty et al., 2022). Additionally, studies have shown that employing contextual problems and forming small groups are effective strategies for activating student learning (Widjaja, 2013; Hofmann & Mercer, 2016). Therefore, IST1's instructional practices align with established pedagogical principles, enhancing student engagement and learning outcomes in permutation and combination topics.

IST2 demonstrated a commitment to self-reflection on teaching practices and a willingness to explore avenues for improvement. Recognizing the importance of a thorough understanding of basic permutation and combination concepts, IST2 endeavored to connect these concepts coherently and structure their teaching approach accordingly. This was evident through reflections on the learning process and focusing on enhancing concept delivery, such as initiating discussions on problems at the outset of lessons. IST2 also prioritized identifying sources of student misunderstandings, promptly rectifying errors, engaging students through open-ended questioning, and addressing their difficulties effectively.

Furthermore, teachers, including IST2, recognized the value of incorporating various learning approaches and representations to support student understanding of permutation and combination material. This finding underscores the importance of considering students' thinking as a crucial learning resource, as understanding their misconceptions can make learning more meaningful. These insights align with research by Soto et al. (2022), which emphasizes the centrality of students' thinking in supporting the development of relevant concepts in combinatorics teaching. However, IST2 has yet to fully integrate technology into the learning process, preferring to base learning methods on the established learning objectives and student needs. This approach aligns with research by McCulloch et al. (2018), which suggests that teachers carefully consider the alignment of technology with learning objectives before incorporating specific tools into their teaching practices.

IST3 experienced significant improvement during collaborative learning activities within the PLCs. Initially, IST3 needed help with planning instructional content, mainly due to the interconnected nature of permutation and combination concepts. Attempting to present both concepts in a single session led to challenges in managing subsequent content presentation timing during teaching practices. As a result, IST3 opted for separate presentations of the concepts to ensure students' strong understanding.

Furthermore, IST3 paid close attention to mentor feedback, aiming to enhance instructional materials' engagement using tools like the book creator. Teachers, including IST3, recognized the imperative for improvement in teaching practices, especially in addressing student misconceptions and enhancing the learning process. This observation aligns with Tam's (2015) findings, which illustrate how the presence of PLCs aids teachers in overcoming initial challenges and boosts their motivation for transformation. The study highlighted several dimensions of change experienced by teachers, including shifts in curriculum, teaching methodologies, learning approaches, teacher roles, and pedagogical practices, as reflected in Figure 4 and Figure 5.

This study offers novel insights that can complement and extend the research conducted by Matitaputty et al. (2022) regarding teachers' knowledge of teaching permutations and combinations. By focusing on teacher knowledge changes during the TPE Program participation, this study sheds light on the dynamic nature of pedagogy and interaction theory. Specifically, the role of PLCs emerges as a crucial factor in supporting shifts in MTSK. Through PLCs, teachers are provided with opportunities to engage in collaborative professional development, facilitating the exchange of ideas, strategies, and best practices.

As depicted in Figure 4 and Figure 5, PLCs serve as a platform for teachers to fulfill their professional responsibilities effectively, fostering an environment conducive to continuous growth and improvement. This highlights the importance of collaborative learning environments in promoting the development of MTSK and enhancing pedagogical practices. Overall, this study contributes to the existing body of literature by emphasizing the significance of PLCs in facilitating changes in teachers' knowledge and pedagogical approaches, thereby enriching the teaching and learning experiences related to permutations and combinations.

CONCLUSION

In conclusion, professional development programs have demonstrated their pivotal role in enhancing the specialized knowledge of mathematics teachers. These programs have not only contributed to significant improvements in teachers' understanding of complex mathematical concepts but have also empowered them to devise innovative teaching strategies. The PLCs have further underscored the potential for collaborative knowledge sharing and experiential learning, fostering higher levels of professionalism and enhancing teacher performance.

While professional development initiatives aim to equip teachers with practical tools for teaching, it is imperative to acknowledge and address the evolving challenges within the educational landscape to enhance the quality of mathematics instruction continually. By cultivating heightened knowledge and expertise, teachers can inspire students and cultivate a deeper understanding of mathematics, thereby fostering a culture of excellence in education.

This study also has certain limitations that should be acknowledged. Firstly, the sample used in the study may only partially represent the diversity of mathematics teachers across various educational levels, regions, and demographic backgrounds. Consequently, the generalizability of the results may be limited. Additionally, the focus of the study was primarily on short-term outcomes immediately following the completion of a professional development program. Therefore, the long-term impact on sustained knowledge improvement and practical application in the classroom needs to be thoroughly explored.

Furthermore, the study should have comprehensively examined external factors such as school culture, administrative support, or curriculum changes, which could influence the effectiveness of professional development programs. Implications for further investigation include the need to delve deeper into the development of mathematics teacher-specific knowledge, particularly regarding teaching permutation and combination topics. Future studies could focus on exploring pedagogical aspects and expanding content knowledge among teachers, especially in combinatorics topics taught at the college level. Lastly, investigating student thought processes and their interference in solving permutation and combination problems could provide valuable insights for enhancing teaching strategies in mathematics education.

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Conflict of Interest	:	The authors declare no conflict of interest.
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