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Classifying analogical thinking for mathematical problem-solving

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Abstract

Analogical thinking is a crucial strategy for mathematical problem-solving, enabling the discovery of solutions by identifying similarities between different problems. However, existing research needs a comprehensive classification of students' use of analogical thinking in this context. This study aims to develop a new classification framework for analogical thinking in mathematical problem-solving, emphasizing the identification and utilization of analogous methods between source and target problems. The research adopts a descriptive qualitative approach involving a purposive sample of 15 high school students from Surakarta, Central Java, who demonstrated analogical thinking in solving both source and target problems. Data collection was conducted through tests, observations, and interviews, with analysis performed using the constant comparative procedure (CCP). The findings reveal three distinct classifications of analogical thinking: pattern recognition (identifying common patterns to solve both source and target problems), variable utilization (using variables as symbolic tools for problem-solving), and visualization (applying graphical representations to address the issues). This study offers significant theoretical insights for future research and practical implications for applying analogical thinking in enhancing mathematical problem-solving.

Keywords: Analogical Thinking, Classification, Constant Comparative Procedure, Mathematical Problem-Solving

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Thinking and reasoning are inherently intertwined, and both are crucial in the process of problem-solving. To effectively address any problem, one must possess the capabilities of both thought and reasoning. For students, the ability to engage in logical thinking and apply mathematical reasoning is vital for solving mathematical problems and exercises (Hidayat et al., 2022; Rahayuningsih et al., 2021; Saleh et al., 2017). The development of these cognitive skills facilitates a more rapid and comprehensive understanding of mathematical concepts. As noted by Fauzi et al. (2020), the ability to think and reason is fundamental in supporting the comprehension of interconnected mathematical concepts.

Analogical thinking is an effective approach to problem-solving, particularly for addressing problems that are ambiguous, novel, or complex (Gentner & Loewenstein, 2002). Employing analogical thinking in the context of mathematical problem-solving can facilitate the discovery of solutions. When an individual uses analogical reasoning to solve mathematical problems, they are effectively identifying relationships between the new problem and their existing knowledge (Lailiyah et al., 2017; Novick &

Holyoak, 1991; Vybihal & Shultz, 1989). Consequently, problem-solving through analogical thinking is often more efficient, as it leverages pre-existing information or knowledge.

Analogical thinking represents a form of creative cognition. It serves as a catalyst for enhancing creativity by introducing new concepts into a particular domain through analogical transfer (Gust et al., 2008; Mutia et al., 2023). The more frequently an individual engages in solving mathematical problems through analogies, the more robust their analogical thinking processes become, which can subsequently be applied to solving real-world problems (Azmi, 2017).

Analogy is a foundational element of human intelligence (Gentner, 2010; 2016; Penn et al., 2008), as it involves the alignment of relationships between current situations and those retrieved from memory (Eveleth, 1999). Analogical thinking is utilized to infer solutions to new problems based on previously encountered ones (Gentner, 1983). This view is consistent with the argument of Mutia et al. (2023), who assert that analogical thinking is a cognitive process where new insights are derived from familiar concepts by identifying similarities between them. In the context of mathematics, analogical thinking is applied to solve diverse mathematical problems by analyzing their underlying similarities.

Ruppert (2013) delineates several stages of analogical thinking in the context of solving mathematical problems: structuring, mapping, applying, and verifying. The structuring phase involves identifying each mathematical object within the target problem by coding its characteristics and drawing conclusions based on the relational similarities between the source and target problems. During the mapping stage, identical relationships between the character codes of the source and target problems are identified, allowing for inferences to be drawn based on these similarities. This relationship is subsequently mapped onto the target problem. The applying stage entails solving the target problem using the steps derived from the source problem. Finally, the verifying stage involves checking the solution to the target problem by assessing the consistency and compatibility between the source and target problems.

Problem-solving through analogical thinking involves leveraging the solving structure of a known problem (the source problem) to assist in addressing a new, related problem (the target problem) (Lailiyah et al., 2022; Novick, 1988). According to English (1999), the source problem is typically presented prior to the target problem and is characterized by its relative simplicity, often being easy or moderately challenging. The source problem can either facilitate the resolution of the target problem or serve as foundational knowledge for approaching the target problem (Kristayulita, 2021). In contrast, the target problem is usually a modified or expanded version of the source problem and is characterized by its complexity (English, 1999). The structure of the target problem is inherently related to that of the source problem (Kristayulita et al., 2020).

In the process of analogical thinking, students exhibit variations in their approaches to solving the target problem. These differences highlight the need for research aimed at classifying students' analogical thinking strategies when addressing target problems. Furthermore, researchers have categorized studies on analogical thinking into three primary areas: analogical thinking in solving mathematical problems, dimensions of analogical thinking, and analogical thinking in mathematics education.

Research on analogical thinking in mathematical problem-solving has yielded several key insights. First, students with varying levels of mathematical disposition—whether high, medium, or low—tend to exhibit similar analogical thinking processes when solving problems. Mathematical disposition refers to an attitude characterized by hard work, perseverance, and confidence in mathematics (Mutia et al., 2023). Second, while students with high and medium abilities typically navigate the encoding, mapping, and applying stages of analogical thinking, the inferring stage is accessible to students across all ability levels

(Purwanti et al., 2016). However, students may make errors in one or more components of analogical thinking—structuring, mapping, applying, or verifying—while solving mathematical problems (Kristayulita et al., 2018). Additionally, Saleh et al. (2017) identified five types of errors students might make when solving math problems, based on Newman's (1977) procedure: reading errors, misunderstanding errors, transformation errors, skill process errors, encoding errors, and careless errors.

In the literature, researchers have also explored the dimensions of analogical thinking, including analogical thinking scheme models, types of analogical structuring, and the application of analogical reasoning in proving mathematical theorems. Kristayulita et al. (2020) proposed two analogical thinking scheme models for solving analogy problems: one for problems that are similar, and another for problems that differ between the source and target. For similar problems, students directly map the procedures from the source problem onto the target problem, facilitating direct information transfer between the two. For different problems, students do not directly map the source problem's procedure onto the target problem and require an additional stage before engaging in analogical thinking. Lailiyah et al. (2018) further distinguish between two types of analogical structuring: internal structuring, which focuses solely on the source problem, and connective external structuring, which links elements between the source and target problems. Analogical reasoning also plays a role in proving theorems, where the proof process involves identifying similarities between the definitions and theorems of the source and target problems (Hardiani & Kristayulita, 2023).

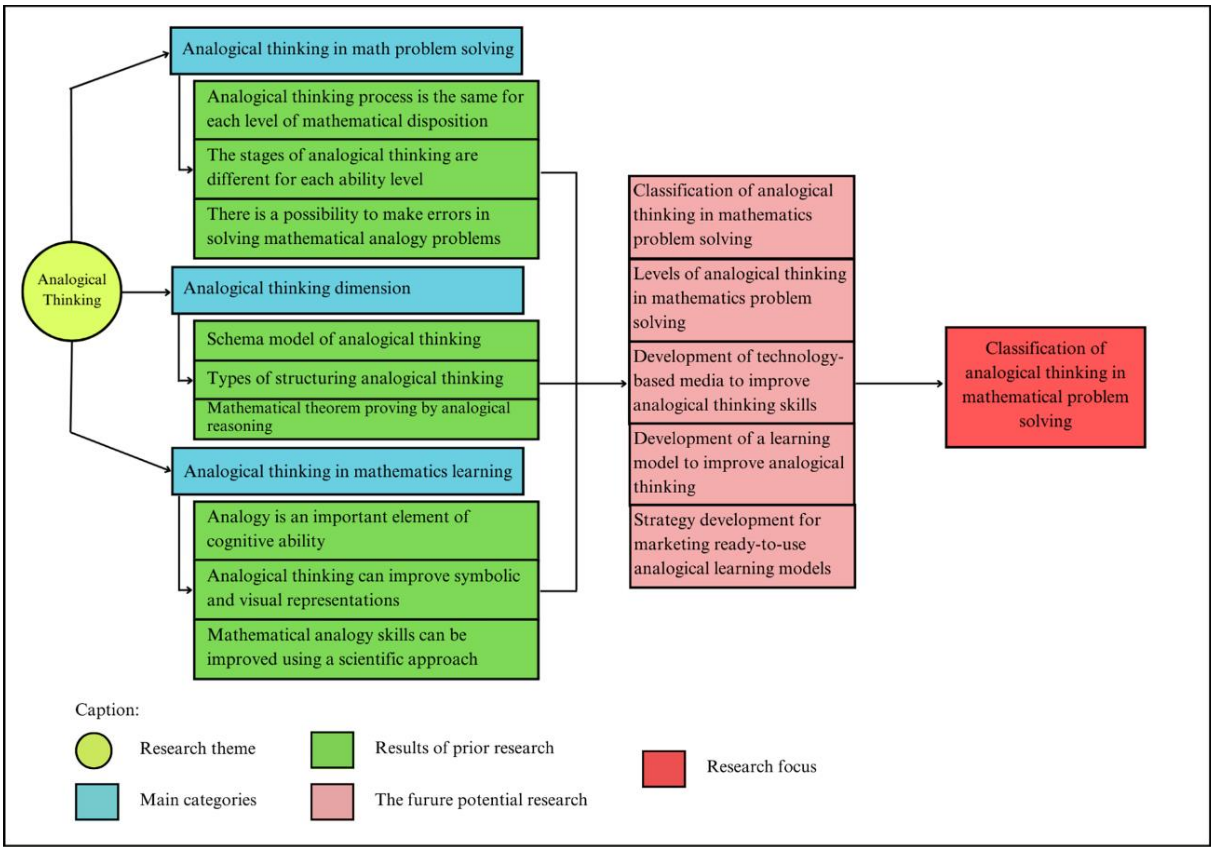


Figure 1. The Research Positions

Researchers have also investigated the role of analogical thinking in mathematics education. Analogy plays a crucial role in various cognitive functions, including memory retrieval, adaptation, learning, reasoning, and creativity (Gust et al., 2008). In the context of geometry learning, analogies are classified into several types: analogies used to understand and establish geometric concepts, analogies relating to concepts, theorems, and properties, analogies applied to problem contexts, problem-solving through analogy using fundamental theorems or general methods, and the formulation of mathematical results by considering analogies (Magdaş, 2015). Analogical thinking is recognized as an effective pedagogical tool in mathematics education (Loc & Uyen, 2014), as it enhances students' symbolic and visual representations. Conversely, learning without analogies tends to improve verbal representations (Dwirahayu et al., 2018). The application of a scientific approach in the learning process can further enhance students' mathematical analogical skills (Hendriana et al., 2018).

Following a review of the literature on analogical thinking, researchers identified several potential research gaps as shown in Figure 1. These include the need for further classification of analogical thinking in mathematical problem-solving, the examination of different levels of analogical thinking, the development of technology-based tools to enhance analogical thinking skills, the creation of instructional models to improve analogical thinking, and the formulation of strategies for marketing ready-to-use analogical learning models. This year, research has particularly focused on classifying analogical thinking in mathematical problem-solving.

Based on a comprehensive literature review conducted in 2023, researchers have focused on classifying analogical thinking in mathematical problem-solving. The findings from this review are intended to guide subsequent research efforts. The research agenda for 2024 aims to investigate the levels of analogical thinking in mathematical problem-solving. In 2025, the focus will shift to developing technology-based tools to enhance analogical thinking skills. The research plan for 2026 will concentrate on creating instructional models designed to improve analogical thinking. Finally, in 2027, the goal is to develop a marketing strategy for a ready-to-use analogical learning model. On the other hand, researchers have conducted numerous literature reviews on analogical thinking and plan to continue exploring this topic in the coming years. Figure 2 illustrates this research roadmap.

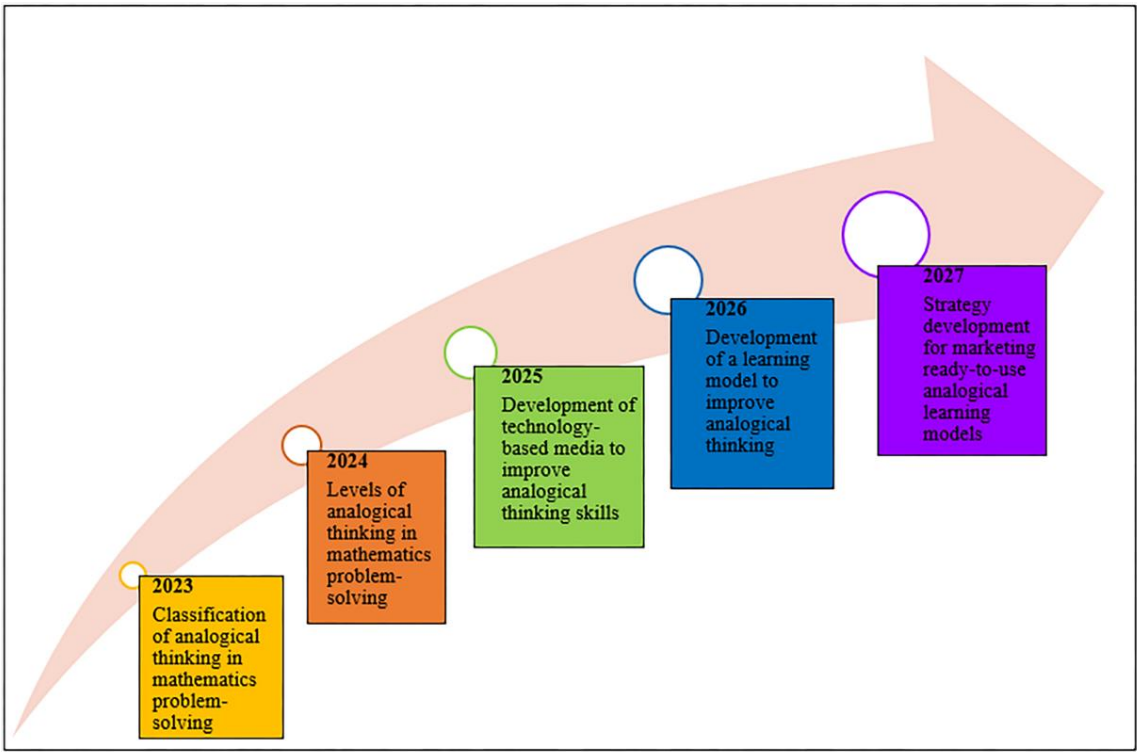


Figure 2. Research Roadmap

This study aims to explore the classification of students' analogical thinking in solving mathematical analogy problems, specifically focusing on source and target problems using the stages of analogical thinking: structuring, mapping, applying, and verifying. The objective is to categorize students' analogical thinking according to these stages when addressing mathematical analogy problems.

The research paper will address how to classify students' analogical thinking based on these four stages in solving mathematical analogy problems. Analogical thinking significantly enhances individual performance in problem-solving (Gentner et al., 2001; Gentner & Loewenstein, 2002), and this study represents an effort to improve such performance.

Studying the classification of students' analogical thinking in mathematics problem-solving is crucial. The results of this study will contribute to theoretical frameworks for future research aimed at enhancing students' analogical thinking skills through techniques such as defragmenting or scaffolding. These techniques involve reorganizing the components and indicators of analogical thinking within individual cognition. This research aligns with the focus of educational research in Indonesia, which is guided by the National Research Master Plan (RIRN). This plan emphasizes themes related to social humanities, cultural arts, and education from 2017 to 2045, encompassing aspects of education and culture. Figure 3 illustrates the research focus.

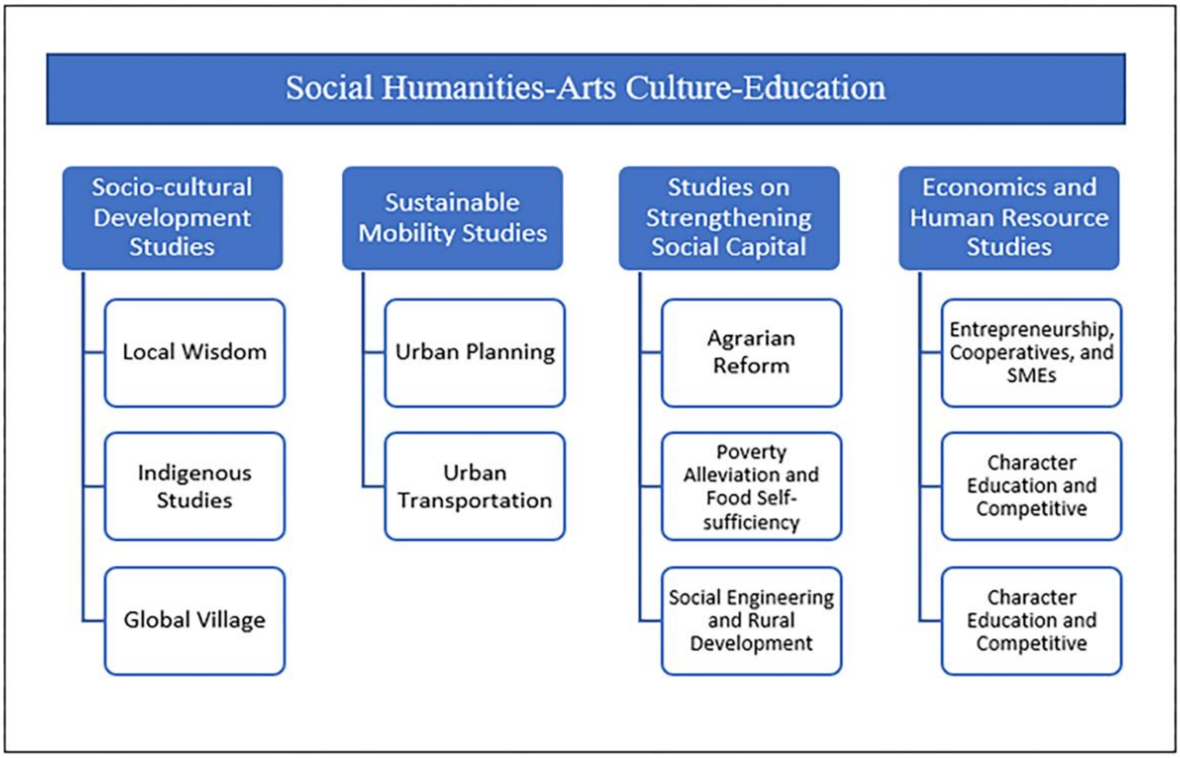


Figure 3. Research Focus: Social Humanities, Cultural Arts, and Education

METHODS

Research Design

This research employs a qualitative, descriptive approach, which is suited for elucidating the results of exploring social processes or phenomena (Creswell, 2014). The descriptive approach involves systematic procedures for data analysis and guidelines for synthesizing textual and structural descriptions. Based on data from the first and second research questions, analysts review the data (e.g., interview transcriptions) and identify "significant statements" that elucidate how participants experience the phenomenon (Creswell & Poth, 2007). Given the objective of this research—to describe the classification of analogical thinking in mathematical problem-solving—and considering that the data is qualitative, the chosen methodology is both descriptive and qualitative.

Participants

The subjects of this study were high-school students from various schools in Central Java and Yogyakarta, Indonesia. Using a purposive sampling method, 15 participants were selected due to their use of analogical thinking in solving mathematical problems (Etikan et al., 2016). These participants addressed both the source problem and the target problem using the same solution concepts. They voluntarily agreed to participate by signing informed consent forms. The researchers ensured the confidentiality of the participants' identities throughout the data collection and presentation process, in accordance with ethical considerations and guidelines set by The Committee on Publication Ethics (COPE). Table 1 illustrates the distribution of participants across the different cities.

Table 1. Demographic of Subjects

City of Origin	Number of Subject
Surakarta	4
Yogyakarta	6
Semarang	5
Sum	15

Instrument

The primary instrument for this research is the researcher, who serves as the planner and data collector, responsible for interpreting, analyzing, and concluding data, as well as reporting the research results (Authors). The auxiliary instruments include mathematical analogy problems, consisting of a pair of problems: one source problem and one target problem. Additional tools are observation sheets to assess students' analogical thinking in solving mathematical analogy problems and audio-visual recording devices.

The instruments were validated by experts specializing in mathematical thinking research and qualitative research in mathematics education. To minimize bias, the instruments underwent piloting with a non-participant or trial group. Feedback from this trial group was used to refine and clarify the problem statements prior to their implementation in the study. The mathematical analogy problems used involved function material for the source problem and comparison material for the target problem. The function problem, being relatively straightforward, serves as the source problem and aids in solving the target problem. The comparison problem, being more complex and structurally related to the source problem, serves as the target problem (English, 1999). The instruments were aligned with the stages of analogical thinking: structuring, mapping, applying, and verifying (Ruppert, 2013).

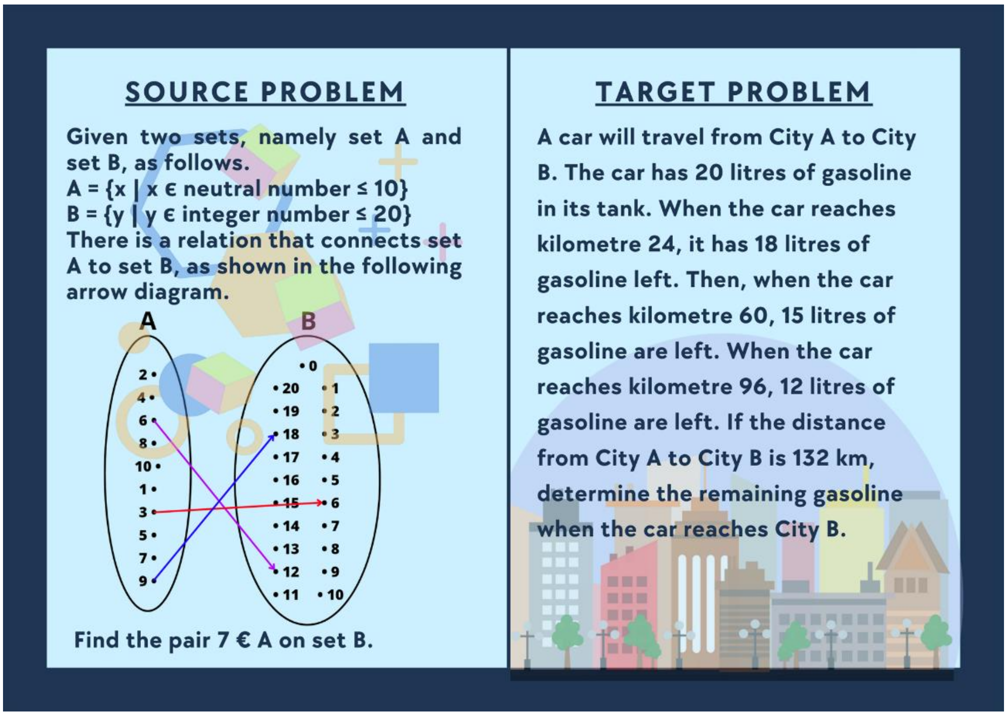


Figure 4. Source Problem and Target Problem to Test Students' Analogical Thinking

Source Problem and Target Problem to Testing Students' Analogical Thinking

Ruppert (2013) explains that the stages of using analogical thinking to solve mathematical problems are structuring, mapping, applying, and verifying. During the structuring stage, the target problem is identified and compared for similarities with the source problem. In the mapping stage, the solution steps of the source problem are systematically aligned with the steps required to solve the target problem. In the applying stage, the solution steps derived from the source problem are utilized to address the target problem. Finally, in the verifying stage, the solution to the target problem is validated by referencing the source problem.

The structure of the source and target problems, as illustrated in Figure 4, can be analyzed according to these stages of analogical thinking. The results of this breakdown are presented in Figure 5.

To solve the source problem, the process begins with identifying the problem by noting the given information. In this case, the problem specifies that pairs of members from set A in set B are as follows: 6 ∈ A pairs with 12 ∈ B, 3 ∈ A pairs with 6 ∈ B, and 9 ∈ A pairs with 18 ∈ B. To find the pair for 7 ∈ A in set B, the concept of proportional comparison is applied, resulting in the answer of 14.

For the target problem, the process starts with structuring: identifying the problem by noting the given information as done in the source problem. The target problem states that the available gasoline supply is 20 liters. The problem specifies that 2 liters of gasoline are used for a 24 km trip, 5 liters for a 60 km trip, and 8 liters for a 96 km trip, with the distance from city A to city B being 132 km. In the mapping process, the solution steps from the source problem are applied to address the target problem. During the applying stage, these steps are used to determine the amount of gasoline required for a 132 km trip, yielding an answer of 11 liters. Finally, in the verifying stage, the solution is checked against the source problem. Since the question asks for the remaining gasoline upon arrival in city B, this is calculated by subtracting the gasoline used from the initial supply: 20 liters – 11 liters = 9 liters.

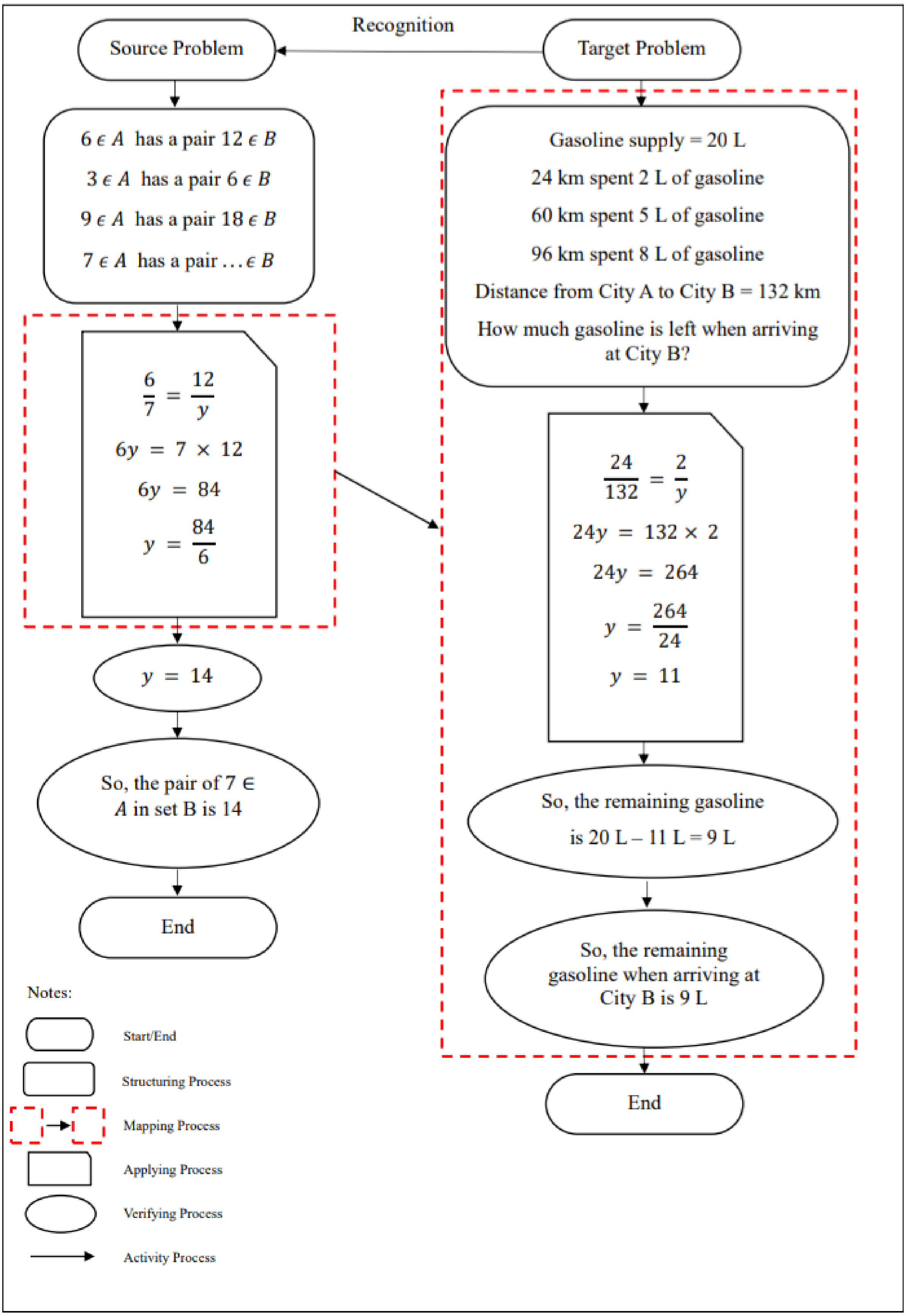


Figure 5. Structure of Source Problem and Target Problem Based on Stage of Analogical Thinking

Data-Collection Procedure

The data-collection procedures were conducted as follows: (1) The researcher prepared the research instruments, which included mathematical analogy problems in the form of source and target problems, field note sheets, and recording devices; (2) Students were provided with the mathematical analogy problems; (3) Students were instructed to solve the problems using creative problem-solving methods; (4) The researcher reviewed and checked the students' solutions; (5) The researcher transcribed the students' written responses to the problems; (6) Interviews were conducted based on the students' answers to uncover their analogical thinking processes; (7) The interviews were transcribed; and (8) The researcher documented both the students' answers and the interview results. The data collection procedure is illustrated in Figure 6.

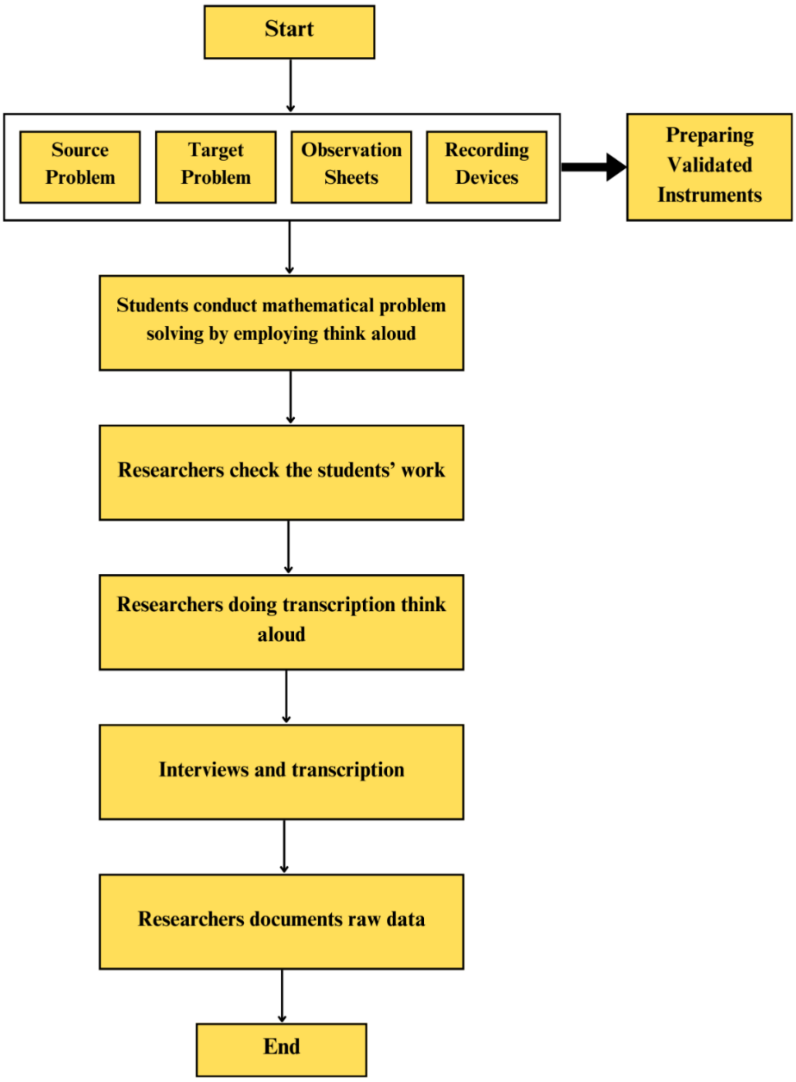


Figure 6. Data collection procedure

Data Analysis

The researcher employed the constant comparative procedure (CCP) for data collection and analysis, as described by Creswell (2014) and Glaser (1992). This inductive data-analysis method involves progressively developing categories of information by comparing data incidents to one another, incidents to categories, and categories to each other. CCP, which moves from specific observations to general conclusions, was applied in this study to transform raw data from student answer sheets, observation sheets, and in-depth interview transcripts into the stages of analogical thinking: structuring, mapping, applying, and verifying.

Data were collected from various sources, including different individuals, multiple sources, and over time from the same individuals. The researcher categorized the data based on analogical thinking methods, such as pattern recognition, variable usage, or visual approaches. Throughout this process, the researcher continuously compared stages and categories to minimize redundancy and strengthen the evidence for each category. Figure 7 illustrates the CCP procedure used in this study.

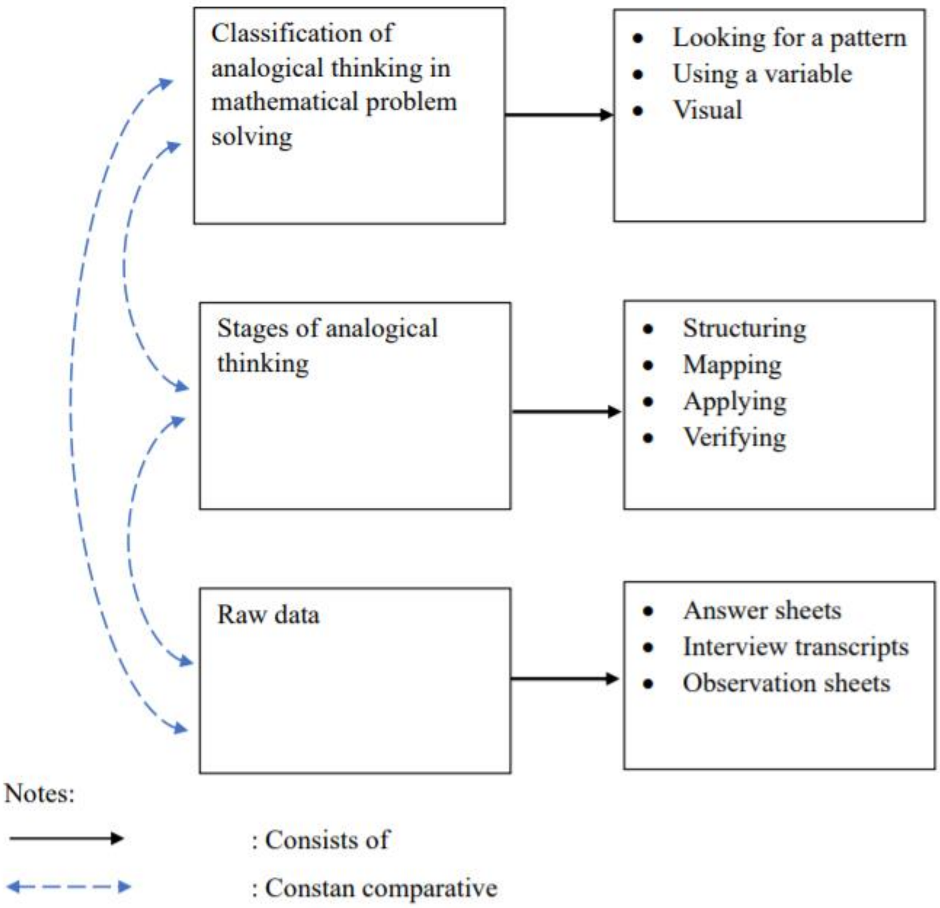


Figure 7. Implementation of CCP in Analysis Data

RESULTS AND DISCUSSION

The results indicate variability in students' analogical thinking processes, which can be categorized based on consistent and general characteristics observed. This section presents data highlighting the consistent and general traits associated with each classification of analogical thinking. These traits, identified through analysis, are maintained as defining characteristics for each classification. In summary, students' analogical thinking is classified according to these consistent and general traits into three categories: pattern recognition, variable usage, and visual approaches. The findings are illustrated in Figures 8–13.

Classification I: Looking for a Pattern

Figure 8 is subject S1's answer to the classification of students' analogical thinking for type 1 in solving the source problem. To solve the source problem, S1 wrote down the members of set A and set B. Furthermore, S1 connected each member of set A with a member of set B based on what was known in the problem using a curved line as a connector, 3 ∈ A is connected with 6 ∈ B, 6 ∈ A is connected with 12 ∈ B and 9 ∈ A is connected with 18 ∈ B. At the bottom of each line, S1 connected the members of set A with the members of set B. At the bottom of each line connecting members of set A with members of set B, S1 wrote ×2, indicating that S1 believes that the relationship of set A with set B forms a pattern that is each member of set A multiplied by 2, then produces members in set B. In determining the pair 7 ∈ A in set B, S1 performed the multiplication operation, which is 7×2 = 14. Then S1 wrote the conclusion that the pair for A = 7 is B = 7×2 = 14.

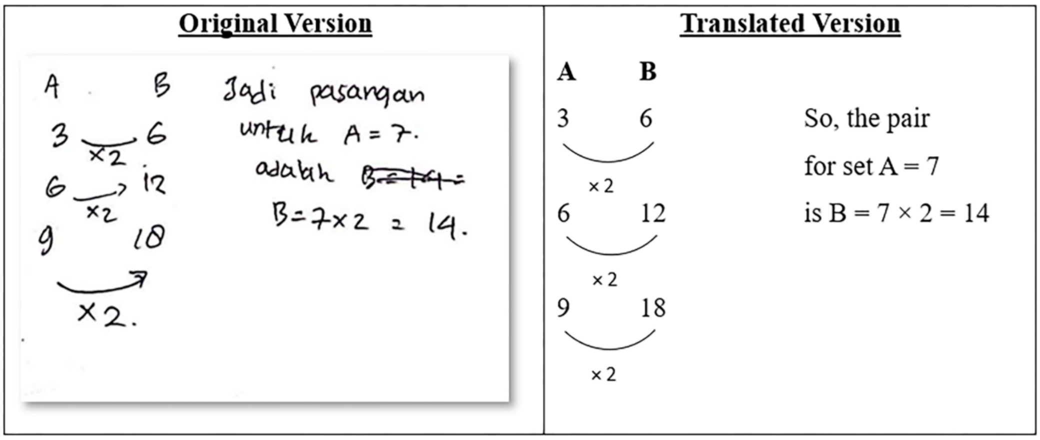


Figure 8. Type I Source Problem Answers

Furthermore, Figure 9 is S1's result of the classification of students' analogical thinking for type 1 in solving the target problem. To solve the target problem, S1 performed the structuring process: identifying the problem as done in solving the source problem by writing the distance and gasoline used to travel the distance. Furthermore, S1 connected each distance with the gasoline used based on what was known in the problem using a curved line as a link as done in the source problem, namely 24 km connected with 2 liters, 60 km connected with 5 liters, and 96 km connected with 8 liters. At the top of each line connecting the distance with the gasoline used, S1 wrote "×12", indicating that S1 believes that the distance with the gasoline used forms a pattern: if the gasoline used is multiplied by 12, then it will result in the distance that can be traveled by using the gasoline. Furthermore, S1 also performed the mapping process, mapping the solution steps in the source problem to solve the target problem by seeing the similarity based on the pattern formed. Then, in the applying process, S1 applied the steps of solving the source problem to the target problem by looking at the pattern formed to determine the gasoline used on the way from city A to city B. S1 wrote that, if the distance is 132 km, the gasoline used is 11 liters because 12×11 = 132. In the verifying process, S1 checked the answer and wrote the conclusion to the target problem. Because what was asked was the remaining gasoline when arriving in city B, S1 then performed a subtraction operation, subtracting from the initial gasoline supply the gasoline used to travel 132 km: 20 liters – 11 liters = 9 liters.

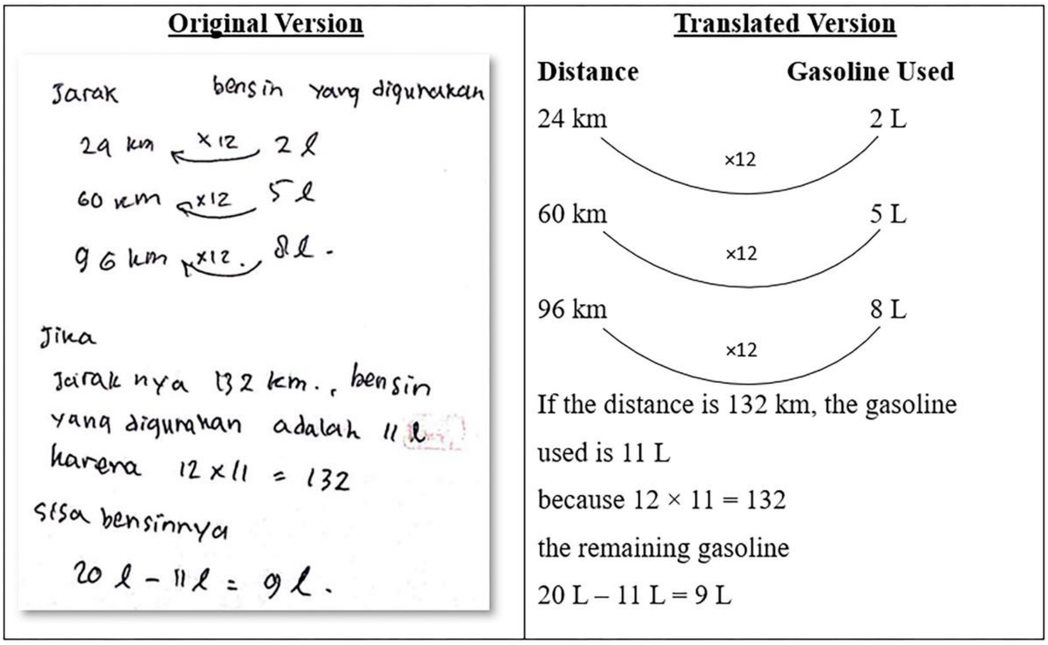


Figure 9. Type I Target Problem Answers

Subject S1 approached both the source and target problems using a consistent solution concept. In solving the source problem, S1 identified a pattern between the members of set A and set B. Similarly, for the target problem, S1 identified a pattern between the distance traveled and the gasoline used. During the problem-solving process, S1 executed all stages of analogical thinking: structuring, mapping, applying, and verifying. This approach demonstrates that S1 utilized analogical thinking by applying the same conceptual method to solve both the source and target problems through pattern recognition.

Classification II: Use a Variable

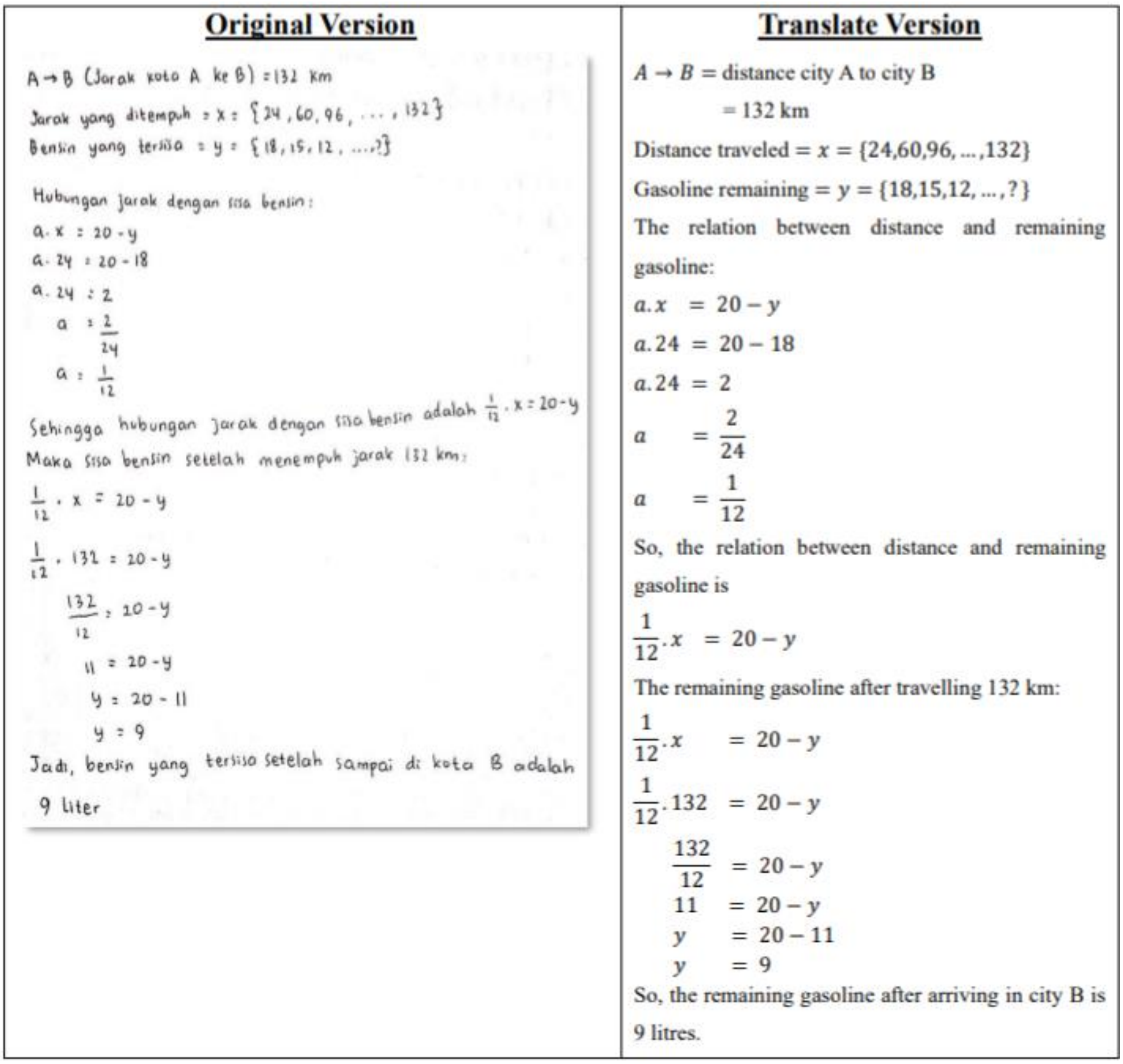
Figure 10 is subject S2's answer to the classification of students' analogical thinking for type 2 in solving the source problem. To solve the source problem, S2 identified the problem first by writing down what was known from the problem. S2 wrote down each member of set A and set B. Next, S2 determined the relationship between set A and set B using a method that uses variables as a formula. The formula is a.x = y, where variable x is a member of set A and variable y is a member of set B, which is a pair of members of set A. To find the relationship between set A and set B, S2 substituted one of the members of set A, 3, as x, along with a member of set B, which is a pair of $3 \in A$, namely 6 as y, into the formula a.x = y so that the formula for the relationship of set A to set B is 2.x = y. Then S2 determined the pair of $7 \in A$ in set B by substituting 7 as x into the formula 2.x = y to obtain the result $y = 14 \in B$, which is the pair of $7 \in A$. Finally, S2 concluded that the pair of $7 \in A$ in set B was 14.

Figure 11 shows subject S2's answer to the classification of students' analogical thinking for type 2 in solving the target problem. To solve the target problem, S2 also did the structuring process, identifying the problem by writing what was known from the problem as done in solving the source problem. S2 wrote that the distance from city A to city B = 132 km, the distance traveled is symbolized by the x variable, and the gasoline used to travel the distance is symbolized by the y variable in sequence according to the information known in the problem. Furthermore, S2 also performed the mapping process stage, mapping the steps of solving the source problem, which involves using variables as a formula to solve the target problem. Then, in the applying process, S2 applied the steps of solving the source problem to the target problem. In the first solution step, S2 determined the relationship between mileage and remaining gasoline using the same method in solving the source problem, which involves using variables as a formula. The formula is a.x = 20-y, where the variable x is the distance traveled and variable y is the remaining gasoline after traveling this distance. To find the relationship between mileage and remaining gasoline, S2 substituted one of the known distances, 24 km as x, and the remaining gasoline needed to travel 24 km, 18 liters as y, into the formula a.x = 20-y so that the relationship formula between mileage and remaining gasoline is $\frac{1}{12}$.x = 20-y. Then S2 determined the remaining gasoline after traveling 132 km by substituting 132 as x into the formula $\frac{1}{12}$.x = 20-y so that the result y = 9 is the remaining gasoline after traveling 132 km. In the verifying process, S2 verified the answer and wrote down the conclusion that the remaining gasoline after reaching city B was 9 liters.

Original Version	**Translate Version**
$A = \{x \mid x \in$ Bilangan asli $\leq 10\}$	$A = \{x \mid x \in \textit{Neutral number} \leq 10\}$
$B = \{y \mid y \in$ Bilangan cacah $\leq 20\}$	$B = \{y \mid y \in \textit{Integer number} \leq 20\}$
$A = \{1,2,3,4,5,6,7,8,9,10\}$	$A = \{1,2,3,4,5,6,7,8,9,10\}$
$B = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20\}$	$B = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15, 16,1718,19,20\}$
Relasi A ke B:	Relation of set A to set B:
$a.x = y$	$a.x = y$
$a.3 = 6$	$a.3 = 6$
$a = \frac{6}{3}$	$a = \frac{6}{3}$
$a = 2$	$a = 2$
Sehingga relasi A ke B adalah $2.x = y$	So, Relation of set A to set B is $2.x = y$
Maka pasangan $7 \in A$ pada himpunan B	Then, the pair of $7 \in A$ in set B is
$2.x = y$	$2.x = y$
$2.7 = 14$	$2.7 = 14$
Jadi, pasangan $7 \in A$ pada himpunan B adalah 14	So, the pair of $7 \in A$ in set B is 14.

Figure 10. Type II Source Problem Answers

Subject S2 applied a consistent solution concept to both the source and target problems. For the source problem, S2 utilized variables to derive a relationship formula between the members of set A and set B. Similarly, S2 employed variables to formulate a relationship between the distance traveled and the gasoline required in the target problem. Throughout the solution process, S2 adhered to all stages of analogical thinking: structuring, mapping, applying, and verifying. This approach illustrates that S2 utilized analogical thinking by applying the same methodological approach—using variables to derive formulas—to both the source and target problems.



Original Version

A→B (Jarak kota A ke B) = 132 km

Jarak yang ditempuh = x = {24, 60, 96, ..., 132}

Bensin yang tersisa = y = {18, 15, 12, ...?}

Hubungan jarak dengan sisa bensin:

$a.x = 20 - y$

$a.24 = 20 - 18$

$a.24 = 2$

$a = \frac{2}{24}$

$a = \frac{1}{12}$

Sehingga hubungan jarak dengan sisa bensin adalah $\frac{1}{12}.x = 20 - y$

Maka sisa bensin setelah menempuh jarak 132 km:

$\frac{1}{12}.x = 20 - y$

$\frac{1}{12}.132 = 20 - y$

$\frac{132}{12} = 20 - y$

$11 = 20 - y$

$y = 20 - 11$

$y = 9$

Jadi, bensin yang tersisa setelah sampai di kota B adalah 9 liter

Translate Version

$A \rightarrow B$ = distance city A to city B

= 132 km

Distance traveled = $x = \{24,60,96,\ldots,132\}$

Gasoline remaining = $y = \{18,15,12,\ldots,?\}$

The relation between distance and remaining gasoline:

$a.x = 20 - y$

$a.24 = 20 - 18$

$a.24 = 2$

$a = \frac{2}{24}$

$a = \frac{1}{12}$

So, the relation between distance and remaining gasoline is

$\frac{1}{12}.x = 20 - y$

The remaining gasoline after travelling 132 km:

$\frac{1}{12}.x = 20 - y$

$\frac{1}{12}.132 = 20 - y$

$\frac{132}{12} = 20 - y$

$11 = 20 - y$

$y = 20 - 11$

$y = 9$

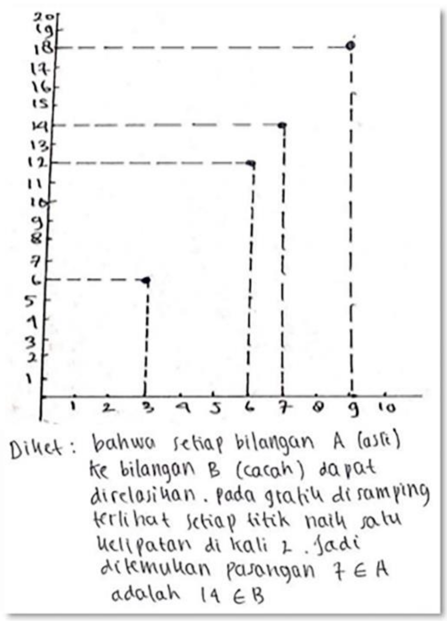
So, the remaining gasoline after arriving in city B is 9 litres.

Figure 11. Type II Target Problem Answers

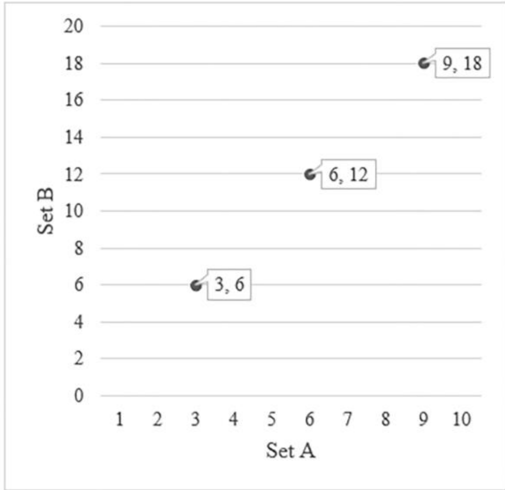
Classification III: Visual

Figure 12 is subject S3's answer to the classification of students' analogical thinking for type 3 in solving the source problem. To solve the source problem, S3 wrote down the members of set A and set B using visualization in the form of a graphic. On the graph, the horizontal part is the members of set A as x, and the vertical part is the members of set B as y. Furthermore, S3 made the coordinate point (x,y) by connecting each member of set A based on its pair on members of set B as known in the problem: $3 \in A$ connected with $6 \in B$ obtained coordinate point (3,6), $6 \in A$ connected with $12 \in B$ obtained coordinate point (6,12), and $9 \in A$ connected with $18 \in B$ obtained coordinate point (9,18). Then, in solving the problem, S3 wrote that every number A (original) and number B (counting) could be correlated. On the graphic, S3 saw each point increase multiple of 2. Thus, S3 considered that each member of set A had a pair in set B, which is two times the members of set A produce their pairs in set B. So S3 can determine the pair $7 \in A$ is $14 \in B$. Then S3 wrote, "So, found the pair $7 \in A$ is $14 \in B$" as a conclusion.

Original Version



Translated Version



It is known: That every number A (natural) to number B (integer) is relatable. In the graph above, it can be seen that each point increases by a multiple of 2. So, the pair $7 \epsilon A$ is $14 \epsilon B$ is found.

Figure 12. Type III Source Problem Answers

Figure 13 is subject S3's answer to the classification of students' analogical thinking for type 3 in solving the target problem. To solve the target problem, S3 did the structuring process, identifying the problem and solving the source problem using visualization, (a graph). The graph illustrates the relationship between the distance traveled and the gasoline used to travel that distance. On the graph, the vertical part is the distance traveled as y, and the horizontal part is the remaining gasoline to travel that distance as x. Furthermore, S3 also did the mapping process stage, mapping the steps of solving the source problem to solve the target problem by describing the graph. Then, in the applying process, S3 applied the steps of solving the source problem to the target problem with the help of a graph made to determine the gasoline used on the way from city A to city B. In the solution process, S3 made the coordinate point (x,y) as done in the source problem by connecting the mileage and the remaining gasoline to travel the distance as known in the problem: the remaining 18 liters of gasoline after traveling 24 km obtained the coordinate point (18,24), the remaining 15 liters of gasoline after traveling 60 km obtained the coordinate point (15,60), and the remaining 12 liters of gasoline after traveling 96 km obtained the coordinate point (12,96). Based on the graph, S3 saw the regularity formed from one of the known mileages: if the mileage is 24 km = remaining gasoline 2 L, then every 12 km uses 1 liter of gasoline. Then, S3 could determine the gasoline used if traveling 132 km by writing that traveling 132 km consumes 11 L of gasoline, making the remaining gasoline 9 L. In verifying, S3 added a coordinate point (11,132) to the graphic, meaning that 11 liters of gasoline can allow 132 km to be traveled. In addition, S3 also verified the answer by writing down how the remaining 9 L of gasoline was found: 20 L – 11 L = 9 L, which means that after traveling from city A to city B for 132 km, the gasoline remaining is 9 liters.

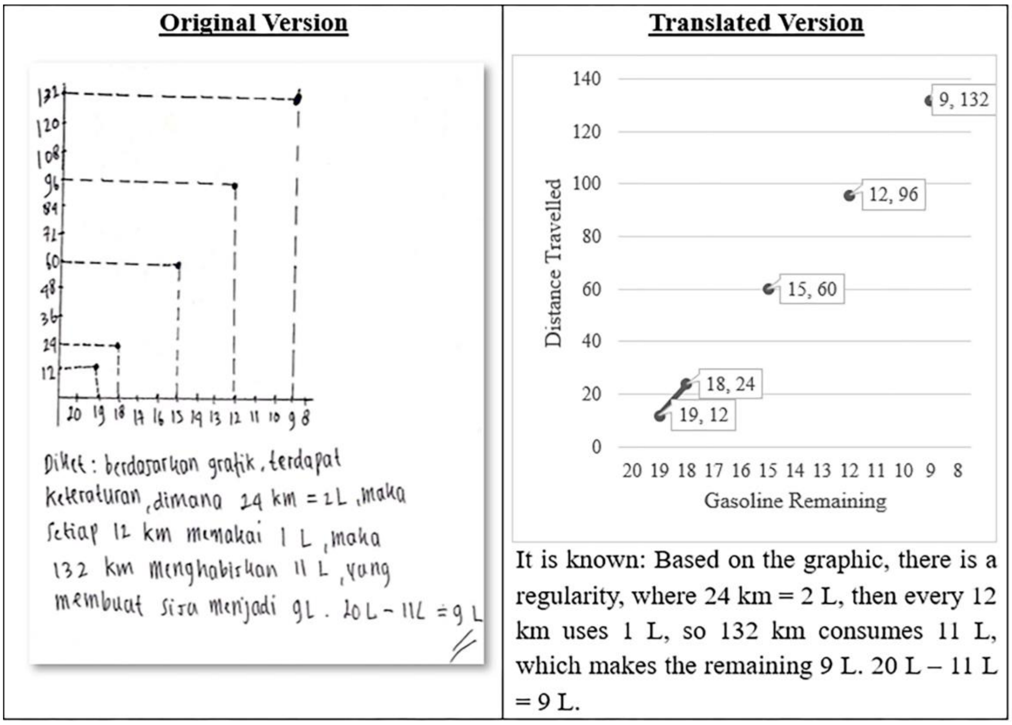


Figure 13. Type III Target Problem Answers

Subject S3 applied a consistent solution method to both the source and target problems. For the source problem, S3 constructed a graph to illustrate the relationship between the members of set A and set B based on the provided information. Similarly, S3 addressed the target problem by creating a graph to represent the relationship between the distance traveled and the remaining gasoline required, using the data given in the problem. Throughout the problem-solving process, S3 completed all stages of analogical thinking: structuring, mapping, applying, and verifying. This approach demonstrates that S3 employed analogical thinking by utilizing the same solution method—visualization through graphing—for both the source and target problems.

Discussion

It is observed that high-school students from various schools in Central Java and Yogyakarta, Indonesia, generally employed all stages of analogical thinking based on the data analysis results. However, some stages were not fully realized. Only 15 students consistently applied analogical thinking, highlighting that its use in education is still underdeveloped. As noted by Hardiani and Kristayulita (2023), the application of analogical thinking in education remains insufficient, with many students either passively or inadequately engaging in this cognitive process. For these students, the primary focus is on providing correct answers rather than understanding the procedural stages of problem-solving (Saleh et al., 2017).

Research by Handayani et al. (2020) suggests that students often lack comprehension of the problem's objectives, failing to leverage their ideas and experiences effectively (Kholid, Hamida, et al., 2020; Kholid et al., 2021). Furthermore, Laurillard (2016) also highlighted that students struggle to fully integrate their knowledge and experience into problem-solving, which impedes their conceptual understanding of mathematical problems (Hidayati et al., 2020). The findings of this study show that students who solved both the source and target problems using the same method demonstrated analogical thinking. According to Saleh et al. (2017), analogical problems share similarities, but students must also consider differences, including the concepts involved, solution procedures, or other aspects.

To effectively use analogical thinking, students need to understand the problem, control and monitor their cognitive processes (Masduki et al., 2020).

This research identifies three distinct classifications of students' analogical thinking in solving mathematical problems:

1. Classification I: Students solve the source problem by identifying patterns and apply the same method to the target problem. They use pattern recognition for both problems.
2. Classification II: Students use variables to solve both the source and target problems, employing algebraic representation to find relationships between elements.
3. Classification III: Students solve the source problem by visualizing it graphically and apply the same visualization approach to solve the target problem.

These classifications reflect different strategies students use in analogical thinking to solve mathematical problems, each demonstrating distinct methods of problem-solving.

Students approach both the source and target problems using a pattern-based solution method in the first classification. They identify predictable or repeatable patterns to solve mathematical problems (Kholid et al., 2022). Pattern recognition is a strategic method for solving mathematical problems, as patterns help individuals perceive order and harmony in seemingly disorganized situations, recognize relationships, and make generalizations (Rahayuningsih et al., 2021). Utilizing patterns allows students to predict relationships from problem information without necessarily resorting to algebraic representations, focusing instead on regularities (Cai, 2003). While pattern-based problem-solving can be efficient, incorporating additional analytical methods could enhance problem-solving efficacy (Sutama et al., 2022).

Furthermore, in the second classification, students use variables to solve both the source and target problems. Variables are crucial tools for systematically describing relationships between two entities (Usiskin, 1999). According to the National Council of Teachers of Mathematics (NCTM, 2020), students should represent unknown quantities using letters or symbols. In this classification, students apply variables to represent and find relationships between different sets or quantities in both problems. For example, students use variables to denote members of sets or components of a mathematical scenario, thereby simplifying problem-solving (Musser et al., 2008; Kholid et al., 2024). The use of variables facilitates a clearer understanding of relationships and enhances problem-solving capabilities (Imroatun et al., 2014).

Finally, in the third classification, students solve both problems through visualization, specifically using graphs. Drawing or visualizing problems is a recognized strategy in mathematics education (Hoon et al., 2013; Novotná et al., 2014). Effective visualization of a problem depends on the student's comprehension of the problem's structure (Krawec, 2014). Students in this classification successfully represent the problem using graphical methods, demonstrating a clear understanding of the problem's context (Kholid, Sa'dijah, et al., 2020; Pradana et al., 2020; Sholihah & Maryono, 2020). Visualization aids in problem-solving by providing a clear, intuitive understanding of the problem, making it a practical strategy (Garderen & Montague, 2003).

CONCLUSION

This study found a new classification of students' analogical thinking in solving mathematical problems. First, the looking for a pattern classification is problem solving by using the same solution concept in the source problem and target problem, namely by looking at the pattern formed from the two problems.

Second, using a variable classification is problem solving by using the same solution concept in the source problem and target problem, namely by using variables as symbols in both problems. Third, visual classification is problem solving by using the same solution concept in the source problem and target problem, namely by visualizing the problem in the form of graphs of the two problems.

The limitation of the research is the number of participants. If the research involves more massive and broad participants, it is possible that the research results will be different. However, this research provides important contributions theoretically for future research and field practice. While contributing to field practice, teachers can implement mathematics learning with differentiated analogical thinking. Outside the research roadmap, future research can focus on the defragmentation of students' analogical thinking at each level in solving mathematical problems and the development of differentiated teaching materials on the classification of analogical thinking.

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Declarations

Author Contribution	:	MNK: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Resources, Supervision, and Writing - review & editing. HF: Data curation, Formal analysis, Project administration, Resources, Software, Visualization, Writing - original draft, and Writing - review & editing. NPL and ICM: Data curation, Formal analysis, Resources, Supervision, Validation, and Writing - review & editing.
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Additional Information	:	Additional information is available for this paper.

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