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Examining undergraduate students' abstraction of conic sections in a dynamic geometry environment

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Abstract

In solid geometry, the concept of conic sections plays an important role in teaching graphs such as parabolas, ellipses, and hyperbolas to undergraduate students in Mathematics Education. It is understood that the abstraction process in mastering conic sections is strongly needed. This study examines the abstraction process of conic sections among third-year undergraduate Mathematics Education students (4 males and 21 females) at Universitas Muhammadiyah Malang (UMM), Indonesia. The data was collected by analyzing students' responses in a 60-minute diagnostic test using the Abstraction in Context (AiC) framework. The test consists of 3 questions, validated by 2 Professors of UMM (average score = 4.14) and 2 lecturers (average score = 4.04). The results showed that 1 male and 11 female students did not reach the construction stage of AiC. Subsequently, a student with a low diagnostic test score and the least completion of AiC stages was observed further through an interview. This student passed through all stages of abstraction with the help of DGE. We also underscored undergraduates' challenges in this process, particularly in visualizing conic section objects, spatial thinking, and employing appropriate mathematical signs. Based on these findings, further research with a broader sample is needed to explore diverse abstraction processes.

Keywords: Abstraction, Conic Section, Difficulties, Dynamic Geometry Environment, Undergraduate Students

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Understanding the concept of conic sections is important for mathematics education students in grasping geometry, specifically for teaching graphs such as parabolas, ellipses, and hyperbolas. Moreover, conic sections have wide applications in everyday life, including engineering, architecture, and design. It enables students to comprehend and solve various practical problems involving conic shapes and their real-life applications. For instance, conic sections play a crucial role in modeling real-life problems, such as representing the orbits of planets and natural satellites in the solar system as ellipses, utilizing parabolic motion in projectile modeling, and employing hyperbolas in satellite signal mapping to precisely determine positions. Based on these, mathematics education students need to master conic sections to later teach them in mathematics classes. Mastery of the concept of conic sections includes the abstraction process, and to overcome it there were some obstacles found by the former researcher, that will be described in this section.

A conic is a curve produced by the intersection of a plane with a vertical conical surface (Florio, 2021, 2022; Fried, 2003; 2022; Heath, 1896; Salinas & Pulido, 2017). From these intersections, circle, ellipse, parabola and hyperbola curves are produced, depend on how to intersect the plane to the cone. A vertical cone cut horizontally by a plane will form a circular curve If it is cut vertically, it will form a hyperbola curve, while if it is cut obliquely, it will form an ellipse or a parabola, depending on the slope. On the other hand, a conic is also defined as the locus of points in a plane that has a fixed distance from a point (focus) and from a straight line (directrix) (Centina, 2016; Florio, 2022). This is related to the eccentricity parameter, which is the ratio of the distance between the point at the position and the focus to the distance between the point and the directrix. In its development, the value of eccentricity was not raised in several books on analytic geometry and its learning to meet the need for self-contained of ellipses, parabolas and hyperbolas (Glaister & Glaister, 2006). The two different approaches that explained above, namely the conic as section of the cone and the conic as the locus of the points, often raise several problems in the learning process. That is, the students could not see clear that two approaches refer to the same object (Florio, 2022). This is due to the lack of understanding of students towards the properties of the conic section. Some researchers provide evidence of many difficulties in learning conic material. Students have difficulty understanding the relationship between conic sections and everyday life. Forty-eight Masters of Mathematics Students at the Department of Mathematics and Computer Science of the University of Calabria participated in a study about parabolas using a historical approach to analytical geometry and the GeoGebra website. They were asked to fill out a questionnaire regarding their initial understanding of parabolas and the effectiveness of learning proposed by researchers. The result was students have difficulty understanding the relationship between conic sections and everyday life. Students see conics only in the context of graphics without understanding their geometric and algebraic properties (Florio, 2022).

To overcome the problems in conic learning, students need to reconstruct their knowledge about conic section. So that, students will understand two or more different definitions or concept of the conic, the definitions not referring to different objects but the same object. Concept construction is a shift in focus from acting on known objects to thinking about how those actions become mental objects that can be manipulated. Mental constructions or abstraction are often described in various frameworks, namely: Action-Process-Object-Schema (Dubinsky & Tall, 1991), procedures, processes (Gray & Tall, 1992), and interiorization-condensation-reification (Sfard, 1991). These encapsulation processes emerge in Piaget's theory of reflective abstraction (Pegg & Tall, 2005).

Abstraction is important for students because it allows students to become aware of similarities among their experiences (Skemp, 1987). In Skemp's (1987) opinion, abstraction is the result of abstracting, then later named by a concept. Abstraction also defined by dually a process of drawing from a situation and also the concept (the abstraction) output by that process. It has a multi-modal meaning as process, property, or concept (Gray & Tall, 2007). In learning mathematics, students need abstractions in order to master the content knowledge. However, various literature states that there are some difficulties faced by students in abstracting knowledge. For example, the occur of partially correct constructs in many cases (Ron et al., 2010, 2017). Ron et al. (2017) analyze the learning process of a grade 8 student when solving elementary probability tasks. The result of this study was despite initial success, he encounters unexpected difficulties in a seemingly straightforward task. Using the RBC model for abstraction in context, the researcher traces his epistemic actions to understand his knowledge construction process, and identify certain constructs, termed PaCCs, concealed within his earlier tasks, which shed light on his subsequent challenges. As well as the difficulty of moving abstraction levels in problem solving (Rich et al., 2019; Rich & Yadav, 2020).

Previous researchers state that abstraction is an essential construct in mathematics education. Various works of literature address various forms of abstraction at various levels of learning mathematics, ways to build mathematical concepts, abstraction in classical and modern views, abstraction and scaffolding (Mitchelmore & White, 2004; Ozmantar & Monaghan, 2007). Rich and Yadav (2020) states that further studies regarding abstraction are still needed, especially in classroom learning and the level of abstraction of primary mathematics students. Komala (2018) explains why it is important to know students' abstraction abilities, namely to find out the problems faced by students and determine the types of interventions that can be given. Hodiyanto et al. (2024) also describe the using of RBC+C in investigating the relation of quadrilaterals and stated that further research is required to investigate abstraction among students with varying levels of geometry proficiency and explore abstraction across different topics beside the quadrilateral.

There are many points of views on abstraction in mathematics. Before the concept of Abstraction in Context appeared, abstraction was a transition from the concrete to the abstract. In this case, it managed the similarity between the new object and the existing schema. In this view, abstraction is a process of decontextualization. This differs from findings about abstractions that appear in social, cultural and human activities (Ozmantar, 2005). The same is stated by Hershkowitz et al. (2001), where a set of external factors influences a person's abstraction process, or it can be said that abstraction is a contextual process. Because of this, the author is currently using a new view of abstraction. That is, abstraction is defined as a vertical reorganization activity of previously constructed mathematical concepts into a new mathematical structure (Hershkowitz et al., 2007; Breive, 2022; Budiarto et al., 2017; Dreyfus et al., 2015; Memnun et al., 2019; Memnun et al., 2017). Vertical mathematization represents the process of constructing new mathematical knowledge within mathematics itself and by means of mathematics. Vertical mathematization usually results by reorganizing previous mathematical constructs, interweaving them into a single process of mathematical thought and leading to new mathematical constructs. Dreyfus et al. (2015) characterize the process of abstraction in four epistemic actions: recognizing, building-with, constructing and consolidation (See Table 1).

Table 1. Indicator based on the completion of stages of AiC

Stages of AiC	Indicators
Recognize (R)	Able to draw the desired cone correctly. Able to draw the desired plane correctly. Capable drawing an arbitrarily ellipse correctly. Capable drawing an arbitrarily parabola correctly. Capable drawing an arbitrarily hyperbola correctly.
Building-with (B)	Capable drawing correctly a pair circle-base of vertical cones of which the cusp cones are coincided. Capable drawing the determine plane that intersects the top cone correctly. Capable drawing the determine plane that intersects the bottom cone correctly. Capable drawing the determine plane that intersects the pair circle-base of vertical cones correctly.
Construction (C)	Capable drawing correctly the desired ellipse which is as the intersection of the chosen cone and plane. Capable drawing correctly the desired parabola which is as the intersection of the chosen cone and plane. Capable drawing correctly the desired hyperbola which is as the intersection of the chosen cone and plane. Able to describe how to obtain properly either the desired ellipse, parabola or hyperbola as conic sections.

Abstracting geometry knowledge, especially on conic, can be challenging for students. Sometimes students need help, both from adults and assistance with tools. The use of tools in learning geometry has existed since the appearance of geometry itself. But the form continues to evolve with the times. The use of assistive devices in geometry material itself is commonly known as the Dynamic Geometry Environment (DGE). DGE has been used in mathematics classes since the 1980s. There are many forms of DGE, both in the form of computer applications, and those that can be held in the hand, such as graphing calculators. DGE provides ways of representing and manipulating geometric objects that are difficult to do with paper, pencil and compass (Hollebrands & Stohl Lee, 2011). One of the uses of DGE, namely exploratory epistemic mediation, is to discover the properties or relationships of geometric objects. This function describes situations where technology supports students in discovering new properties or realizing the structure of mathematical objects. Special tools in DGE allow students to dynamically observe patterns and properties of geometric objects and formulate reasonable conjectures (Yao & Manouchehri, 2019).

The use of DGE in the construction of geometric objects has been carried out by some previous researchers (Elgrably & Leikin, 2021; Ng et al., 2020; Turgut, 2019). The aims of their research was to look how two different groups pose the problem through investigation in DGE (Elgrably & Leikin, 2021), compare the effects of two classroom-based technology-enhanced teaching interventions, focusing in inquire the relations among the number of vertices, edges, and faces of prisms and pyramids (Ng et al., 2020), and analyze students' sense-making regarding matrix representation of geometric transformations in (DGE) within the perspective of semiotic mediation (Turgut, 2019). The studies actually talking about how students abstract new geometry knowledge. But they are not investigating it as an abstraction process explicitly. It makes us interest to study more deeply about the process of abstraction using DGE. Subsequently the DGE used in this research was GeoGebra. Priatna et al. (2018) explained that GeoGebra-assisted learning significantly gives better results in the abstraction process compared to conventional learning. Turgut (2019) explained that the drag tool in GeoGebra led to an understanding of the transformation of geometry material. GeoGebra also help prospectives teacher learn mathematical proof informally (Putra et al., 2023).

Based on the background provided, this study aims to examine the abstraction process of conic sections among third-year undergraduate Mathematics Education students at Universitas Muhammadiyah Malang (UMM), Indonesia. Previous research on abstraction in conic sections was not all using the AiC framework. The integration of Dynamic Geometry Environment (DGE) was intended to aid students in constructing knowledge of conic sections. For example, students could utilize DGE's instant features to visualize geometric objects, such as cones, planes through three points, and ellipses, facilitating their understanding. The explicit examination of DGE's role in the abstraction process through AiC has not been addressed by previous researchers, prompting further investigation by the present researcher. In addition to analyzing students' abstraction processes, the researcher aimed to identify the difficulties encountered by students in acquiring the desired knowledge.

METHODS

The so-called Abstraction in Context (AiC) is adapted to examines the abstraction process of 25 participants of 3rd years undergraduate students of Universitas Muhammadiyah Malang Indonesia. Based on Table 1, any students are satisfied the related stages of AiC, if at least 3 of its stages are fulfilled, otherwise is not or partially satisfied the related stages. All participants, 4 males and 21 females had already finished the course of Analytic Geometry in the second year of their study, and the DGE GeoGebra was implemented in this study. Then all participants must solve the 60 minutes diagnostic test related to specific conic section, parabola, ellipse and hyperbola, consist of 3 questions, and it forms a conventionally pen and paper test. Furthermore, to have more data about student's abstraction, the students who required 1) have a low level of diagnostic test, 2) have the least completion of AiC stage chosen to observe further, by an interview. The interview was aimed to confirm student's test response, as well as see how a conditioning can affect the emergence of each stage of abstraction. This activity was carried out directly by the researcher with the help of audio recordings. The recordings then transcribed for analysis based on the RBC model.

The diagnostic test had declared as valid and feasible which is referred to its logical and content, by two professors of UMM with major in mathematics or mathematics education, with average score is 4.14 based on the scale-5 Likert. The test, which is related to its legibility and appropriateness, had also examined by two lectures of Analytic Geometry with average score is 4.04. Furthermore, employing the statistical analysis, the diagnostic test is valid and reliable, with the score validity is 0.859, the reliability coefficient is 0.819; the item difficulty is 0.67 and the item discrimination is 0.68. The diagnostic test shown below.

Problems

Draw double vertical right circular cone, of which the Vertices are coincident. Then,

1. Draw a plane that pass through either the top or bottom of the cone that yield a conic section resulting an ellipse, then explain your answer.
2. Draw a plane that pass through either the top or bottom of the cone that yield a conic section resulting a parabola, then explain your answer.
3. Draw a plane that pass through both of cones that yield a conic section resulting a hyperbola, then explain your answer.

RESULTS AND DISCUSSION

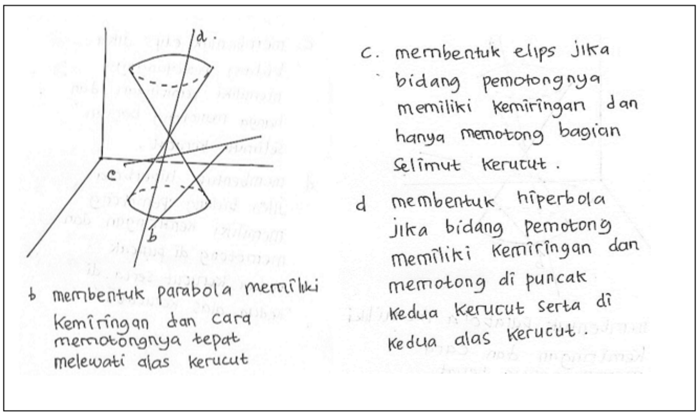
Results

The research results comprise the analysis of the Analytic Geometry scores of 25 students, along with their diagnostic test results and the completion stages of the AiC process. The Analytic Geometry data were obtained from the last Analytic Geometry course, with the result there were 16 students (15 females and 1 male) were in the high category, 9 students (6 females and 3 males) in medium category, and no students in the low category. Moreover, the diagnostic test shows that 7 students (6 females and 1 male) in the high category, 4 students (2 females and 2 males) in the medium category and 14 students (13 females and 1 male) in the low category. Subsequently the analysis of the completion stages of the students were tabulated on Table 2 (**Note:** F (Female Students), M (Male Students)).

Table 2. The Data of The Completion Stages of AiC

Stage of AiC	Number of Students					
	R		B		C	
	F	M	F	M	F	M
Satisfy	19	3	20	3	8	2
Partially Satisfy	2	0	1	1	2	1
Not Satisfy	0	1	0	0	11	1
Total	21	4	21	4	21	4

Based on Table 2, one male student did not satisfy recognize stage, one male and one female student partially satisfied building with stage, and one male did not satisfy construction stage. The male student who always appears at the bottom in Table 2 is the same student, labeled S1. S1 is a male student. In terms of academic ability, S1 is a standout student, as indicated by their GPA that qualifies for the 'very satisfactory' criteria. S1 has achieved A grade in several Algebra courses, while attaining B grades in some Geometry subjects. Notably, in the Analytic Geometry course, S1 has score of Analytic Geometry 81, high category, a result strongly influenced by the predominant algebraic approaches employed within the curriculum of Analytic Geometry. However, S1 has score of diagnostic test 60, low category. S1 is skilled in using GeoGebra. From the Analytical Geometry Course, S1 has acquired knowledge about how the ellipse, parabola and hyperbola refer to a locus of points. The result of S1's diagnostic test was not satisfied recognize stage, partially satisfy building with and not satisfy the construction stage. The interview by the RBC Model is described more comprehensive as shown in Figure 1.



English version:
The parabola b is formed when the plane tilts and intersects the base of the cone.
The ellipse c is formed when the plane intersects the surface of the cone.
The hyperbola d is formed when the plane tilts and intersects the bases of both cones.

Figure 1. The result of S1's test

S1 was able to recognize the cone and draw double cone accurately (See Figure 1). Additionally, S1 demonstrated the ability to draw the cones on the Cartesian Coordinate System but could not draw the conic section. Although he could explain it in a description, he did not draw the conic section showing an ellipse. Furthermore, S1 did not to label each axis and provide adequate information on the image, it indicated from Figure 1. S1 might encounter challenges with spatial thinking and manual drawing techniques, because S1 did not has issues about operating GeoGebra. This inference was drawn from observing Figure 1, where despite being proficient in using GeoGebra, S1's manual drawing skills appeared to be deficient. Consequently, during the interview, S1 was requested to draw manually first, and have a chance to use GeoGebra at the last if needed.

The findings from the interview with S1 will be presented in two distinct sections. The initial section will delve into the elucidation of ellipse concept construction, while the subsequent section will focus on the comprehension of the parabola and hyperbola concepts construction, with the assistance of DGE (Dynamic Geometry Environment).

First Section: Ellipse Concept Construction

An investigation into the student's initial knowledge constructions has been conducted before students did the test. S1 was a student who had completed the analytical geometry course. S1 was introduced to the definitions of ellipses, parabolas, and hyperbolas as loci of points equidistant from the focus and directrix during the course.

The excerpt of interview with the S1 is given as follows:

1	R	:	Could you explain the way you draw this double cone (refer to his test response).
2	S1	:	I will draw a cone with a horizontal base, and the vertex points will coincide. To begin, I need to construct the Cartesian Coordinate system. Next, I draw a circle to represent the base of the cone. Then I establish a point perpendicular to the center of the base circle and connect it with a line segment, which will serve as the height of the cone. (Refer to Figure 2)
3	R	:	Well, is there anything to add to your image?
4	S1	:	I made a mistake in the positioning. There is not sufficient space for drawing the second cone. May I redo the drawing?
5	R	:	Yes, please.
6	S1	:	The dotted line indicates that this line located at the back of the cone, visually differentiating it from the front portion.

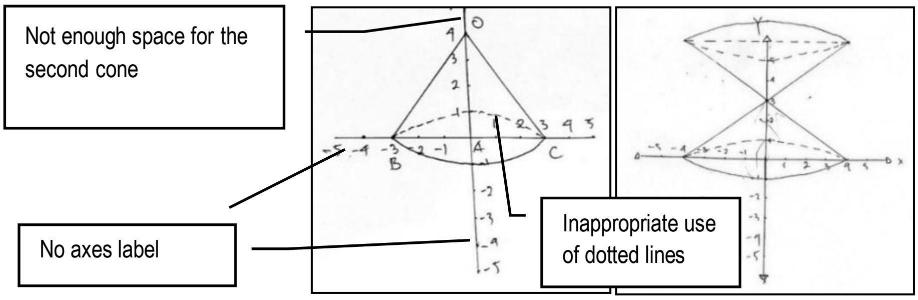


Figure 2. Initial drawing (left), Subsequent drawing (right)

S1 exhibits a clear understanding of the cone concept, as seen from how S1 could draw a double cone on his test response (See Figure 1). This phase of S1's recognition of the old construction represents the initial stage of abstraction, known as the "recognizing" stage. Most other students also recognize the concept of a cone. Moreover, S1 could explain how he drew a single cone during the interview. However, it was worth noting that at the onset of the interview, S1 depicted only a single cone, despite the question explicitly referring to a double cone (See Figure 2a). Therefore, an investigative question is given, to check S1's understanding of the given question (See Row 3). The question made S1 later realize that he made a mistake not considering enough space to draw the next cone. Then, he redrew the requested double cone (See Figure 2b). Furthermore, S1 had not consistently described the coordinate axes. In Figures 2a and 2b, S1 did not draw the z-axis. This is different from the test response, where S1 has been drawn the three axes even without labels (See Figure 1). The images draw by students found in the test results are divided into two types, namely those that draw cones in Cartesian coordinates (type 1), and those that draw cones not in a coordinate system (type 2) (See Figure 3). In the diagnostic test results, there were only two students out of twenty-five students who depicted a double cone in a Cartesian coordinate, these are S1 and S2. The rest of the students belong to type 2. In the test results, S2 also did the same thing that S1 did during the interview, that is only drawing two main axes.

S1's inconsistency in the naming of the axes can be seen from the three figures made by S1. In the test response, S1 omitted the axis labels (See Figure 1), similarly in the initial interview figure, S1 did not label the coordinate axes (See Figure 2). However, in the second interview, S1 accurately label the coordinate axes as the x-axis and y-axis (See Figure 2). Another finding on this occasion was that S1 has an error in using the dotted line on the cone base. This was found in his test response (See Figure

1) and in his subsequent drawing in the interview session (See Figure 2). It indicated that S1 still struggles to mentally visualize the configuration of the double cones accurately. The explanation above proves that S1 has recognized knowledge of cones, the one of last construction that he must have to solve new situation in this problem.

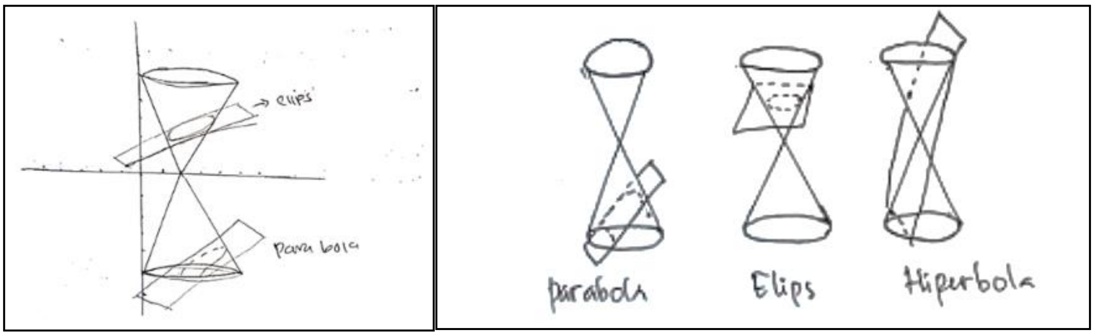


Figure 3. The images draw by students Type 1 (left), Type 2 (right)

The next conversation shows how S1 constructing a conic section from the intersection of a cone and a plane.

7	R	What do you mean by this b, c, and d lines? (Refer to Figure 1)
8	S1	They are planes.
10	R	Are you familiar with the term of a plane?
11	S1	I am still unsure about how to draw the intersection of the plane and the cone.
12	R	Have you ever eaten an "ice cream cone"? If you cut an ice cream cone like this, what shape will the cut be? (Demonstrates cutting ice cream cone by hand).
13	S1	Are you referring to the top part? It will be a cone.
14	R	Not exactly the top, but just the surface. What shape will it be?
15	S1	It's a circle.
16	R	Now, how to get the ellipse?
17	S1	To get an ellipse I need to cut an ice cream cone obliquely.

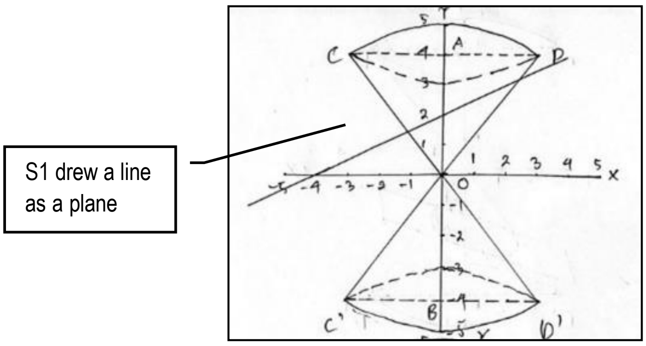


Figure 4. S1 draw a plane as a line

The second knowledge that S1 need to recall was a concept about a plane. S1 seems oblivious to the concept of plane. This was shown in his test response, he represented the planes by the lines. Most students were able to recognize plane, only two people draw the plane as a line on their worksheets, namely S1 and S3. The same thing appeared during the interview. When S1 was asked "what do you mean by this b, c, and d lines?", he replied that the lines were the planes. When S1 asked to redraw, the conic section showing an ellipse, he drew the plane as a line again (See Figure 4). Then researcher employed a concrete example by presenting a piece of paper nearest as an example of a plane S1 was conditioned S1 to recollecting the previously knowledge, the concept of a plane, using a tangible object.

After giving an example of a plane using a sheet of paper, it seems that S1 has been able to recognize the concept of a plane. The next stage that S1 should go through is building with, the second stage in the AiC framework. But S1 still did not know how to draw a conic section showing an ellipse (See Row 11). Twelve students were observed to find the same difficulty in drawing conic section showing an ellipse based on test result. Then the employment of everyday objects like an "ice cream cone" as an analogy was needed to lure the S1 into the building-with stage. By demonstrating how an ice cone would be cut horizontally using hand gestures, S1 has been provided a relatable visual aid. As a result, S1 finally arrived at the correct understanding and identified the shape as a "circle." With S" found a circle as a result of the conic section, this means that S1 has satisfy the third stage of AiC, namely construction. Having a construction about the concept of a circle as a conic section, S1 appeared to find it easier to construct knowledge about the concept of ellipse as a conic section, by imagine cutting the ice cream cone obliquely. Subsequently, S1 was asked to manually draw it on a piece of paper.

18	R	Now that you have grasped the concept of an ellipse as a conic section, please proceed to draw it on your paper.
19	S1	How should I draw the plane? Like this line? (See Figure 4)
20	R	It should resemble a parallelogram. This is how you draw a plane (demonstrates drawing a plane on a piece of paper). Think of it as a parallelogram, and this represents the image of a plane.
21	S1	I am still unsure. I am still finding challenging about draw it on a cone. Is it like this? (Draw a small parallelogram) (See Figure 5)
22	R	That's not entirely accurate. Ok, you can use GeoGebra to make it easier.
23	S1	Alright. First, I need to create the cone by click Tools, then choose Cone. Then I select a point as the center of the circle, choose the next point as the end of the cone, and specify the diameter for the base of the cone (See Figure 6a). To create the section of the plane, I click Tools, then choose the Plane Through 3 Point. I plot the points on the cone to generate an ellipse. However, in my initial attempt, I did not obtain an ellipse shape; instead, I got a parabola (See Figure 6b). Therefore, I replot the points, and now I have successfully achieved an ellipse (See Figures 6c and 6d).

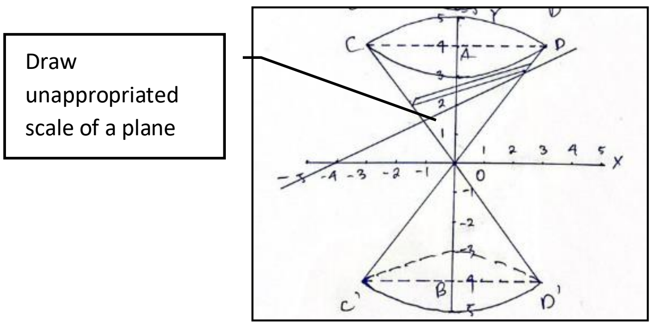


Figure 5. S1 draw a plane with unproportional scale

S1 encountered difficulty in translating his imaginative concepts onto paper. Although S1 could envision ice cream cone being sliced to form circles and ellipses, he struggled to draw them as intersections between cones and planes. Specifically, S1 could draw a cone but lacked the understanding of how to depict a plane intersecting it. His attempt resulted in a plane with only one line segment piercing the cone. As a result, the S1 had been reminded by common representation of plane - a 'parallelogram'. Even though S1 comprehended the visual representation of a plane as a parallelogram, he still faced challenges in illustrating how to intersect it with a cone. This phase aligns with the "building-with" stage of the AiC (Abstraction in Context) framework. During this stage, students utilize previous constructions to build upon their existing knowledge. An additional conditioning had been provided in the form of clues to facilitate this process. The subsequent steps must be taken by the students themselves. In this instance, the clue had been offered regarding the placement of the parallelogram on the cone - specifically, positioning it at the center of the cone. While S1 followed the researcher's instructions, the resulting parallelogram was too small, and there were errors in the use of solid and dashed lines. Consequently, the using GeoGebra were suggested to aid in the construction.

Allowing S1 to utilize GeoGebra proved to be instrumental in advancing him to the second and third stages of the abstraction process, namely "building-with" and "construction." The "building-with" stage was evident as S1 employed GeoGebra to intersect the plane with the cone, effectively utilizing the concepts of the cone and plane (previously learned constructions) to address the new situation. Subsequently, S1 successfully reached the "construction" stage when he obtained an ellipse as the section formed by the intersection of the plane and the cone. S1's activities while utilizing GeoGebra given as follows.

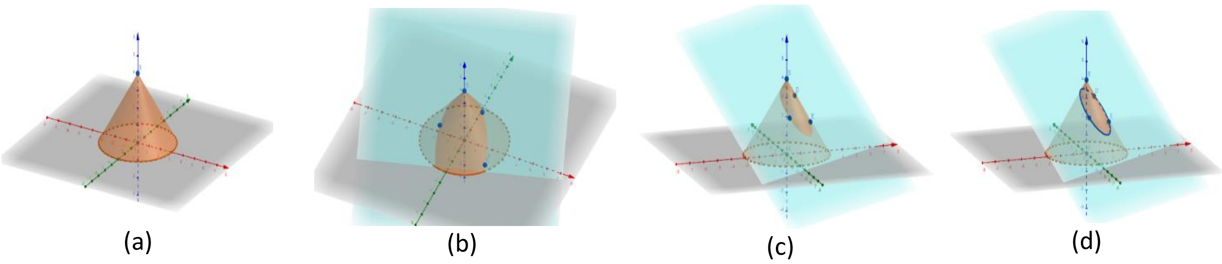


Figure 6. S1 activities using GeoGebra

Following the utilization of GeoGebra, S1 proceeded to attempt drawing the object on paper. As a result of using GeoGebra as an aid, S1 significantly improved his ability to answer the task compared to his previous attempts. However, it is worth noting that he still omitted the representation of the z-axis in his drawing (See Figure 7).

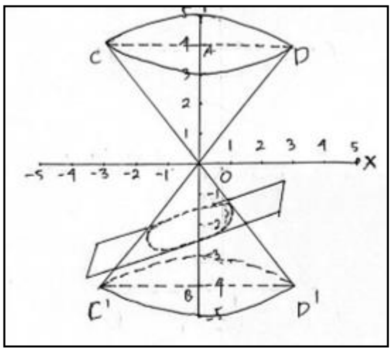


Figure 7. The final answer of S1

Second Section: Construction of the Concept of Parabola and Hyperbola

The S1's knowledge construction about the concept of parabola and hyperbola are explained in this section. For this construction, S1 no longer has as many difficulties as when constructing knowledge about ellipses. A part of conversation that describes S1's construction of knowledge about parabolas and hyperbolas are given below.

24	R	How you cut the cone, so it formed parabola?
25	S1	Is it the parabola? (Draw the parabola outside the cone, See Figure 8)
26	R	Yes, it is.
27	S1	So, this is the parabola (while point to his GeoGebra page, see Figure 6b).
28	R	Please explain how you get it.
29	S1	Like the ellipse, first, I need to create the cones. I click the Tools button, then I choose Cone. Next, I draw the plane using Plane Through 3 Point. I plot one point on the cone frustum, and two points on the base of cone (See Figure 9).
30	R	Ok good, now please draw it on your paper.

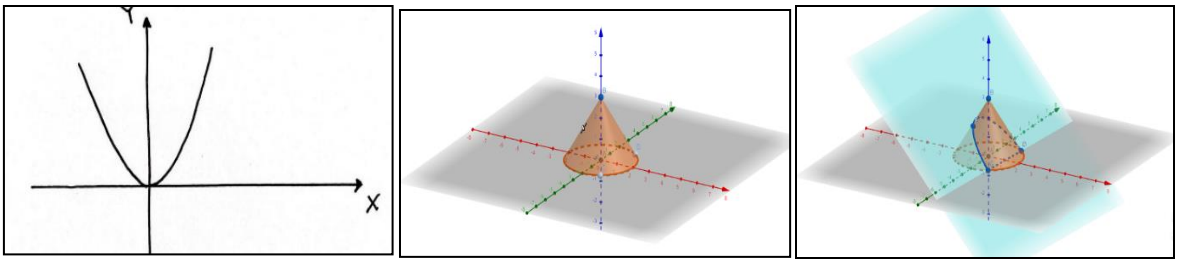
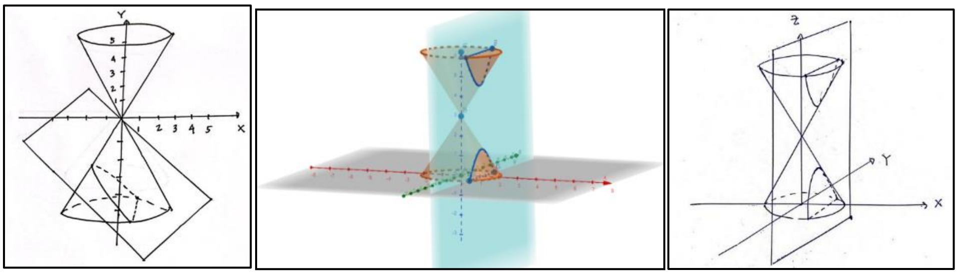


Figure 8. S1 redraw the parabola using GeoGebra

S1 recognized the parabola, but S1 was not sure about knowledge he had. So, he asked the researcher about it, by draw the parabola in Figure 8. On this occasion, S1 no longer need too much help to describe a conic section that forms a parabola. He automatically remembered his last step to draw an ellipse as conical section using GeoGebra. He easily pointed out the parabola shape from some figure that he made. The next he drew the figure on his paper (See Figure 9). In this picture, S1 still ignores the use of dashed lines in some parts of the cone and plane images. After S1 construct the ellipse and parabola knowledge, S1 seems have some insight to draw the conic section showing a hyperbola. So, S1 tried to draw the third construction with the help of GeoGebra firstly. Then he began to draw on the paper. In the result S1 was successful to construct the conic section showing a hyperbola.



(a) (b) (c)

Figure 9. (a) A conic section showing a parabola, (b) A hyperbola construction by GeoGebra (c) A conic section showing a hyperbola

Discussion

The results of the diagnostic test for S1 reveal a score of 60, placing their performance in the lower category. Subsequent completion of the AiC stage underscores that S1 does not attain the "Recognize" stage, as they only fulfil one of the five prescribed recognition indicators. Moreover, S1 only partially aligns with the criteria for the "Building With" stage, satisfying merely one out of the two designated indicators. Lastly, S1 falls short of reaching the "Construction" stage, failing to meet any of the stipulated construction indicators. This aligns with the research conducted by Nurhasanah et al. (2017), which posits that the abstraction process occurring within a group of four prospective teacher students predominantly emphasizes empirical abstraction, focusing on the identification of characteristics of manipulated or imagined objects during the recognizing and building with stages. This contrasts with the findings of (Kouropatov & Dreyfus, 2014), whose research demonstrates that the majority of the eight students succeeded in achieving most of the new constructions in question within a limited timeframe, and many of their construction processes can be elucidated through the RBC model. The analysis reveals that students recognized prior constructs and built upon them during the construction process with minimal researcher intervention.

The issue of partial construction, as identified by the researchers, aligns with the concept of Partially Correct Constructs (PaCCs) proposed by Ron et al. (2010, 2017). Specifically, some of the students' constructs correspond to the mathematical principles underpinning the learning context. In this study, the learning process of eighth-grade students on the topic of basic probability was scrutinized. While students appeared successful in completing initial assignments, challenges emerged when advanced assignments were presented. Notably, these advanced assignments did not require additional competencies beyond those necessary for the initial tasks, yet students encountered difficulties.

During interviews, several difficulties were observed, including challenges in conceptualizing spatial information, articulating mental imagery, and envisaging the outcomes of spatial object manipulation within their minds. These challenges align with previous research, such as: (1) students struggling to engage in the abstraction process when problem-solving, (2) students focusing on the characteristics of observed objects but failing to discern the relationships among them, and (3) students acknowledging the characteristics of observed objects without performing the requisite actions for object manipulation (Dewi et al., 2018). These challenges appear to be particularly pronounced in the context of geometry material, where students are called upon to engage in more intricate geometric and spatial thinking compared to solving algebraic problems. The challenges in spatial thinking are consistent with the findings of Dintarini and Zukhrufurrohmah (2021), who reported that 21% of the 34 students encountered difficulties in thinking spatially while working on Analytical Geometry problems. These challenges were mitigated through the provision of scaffolding using an online format.

During the interview, the researcher employed various strategies to delve deeper into S1's abstraction process. To facilitate this exploration, the researcher utilized several prompts and probes with the aim of encouraging S1 to successfully complete the assigned tasks. It is worth noting that these prompting and probing techniques constitute valuable scaffolding methods, which educators can employ to assist students in their learning journey. According to Munson (2019), these approaches enable instructors to uncover students' thought processes as they develop mathematical concepts and strategies. Furthermore, previous research suggests that ethnomathematics-based prompting and probing have a noteworthy impact on students' communication skills, as evidenced by the findings of (Hartinah et al., 2019). Tutor intervention in the form of reducing uncertainty, directing students' attention, and establishing sub-goals is believed to play a pivotal role in facilitating the subject's knowledge abstraction process.

One of the initial challenges encountered by students pertained to their inability to envision the information provided, specifically, the concept of a "double vertical right circular cone with coincident vertices." As a result, students initially sketched the first cone disproportionately on their answer sheets. To address this challenge, the researcher resorted to probing techniques, inquiring whether the initially drawn image would align with the subsequent requirements of the task. Another challenge that students faced revolved around their difficulty in visualizing flat planes that intersected the cones, giving rise to ellipses, parabolas, and hyperbolas. To mitigate this challenge, the researcher introduced a tangible example by presenting a "concrete object," namely, an "ice cream cone," which was then physically cut with a knife.

During the interview, the researcher strategically employed a range of methods to gain deeper insight into the abstraction process of S1. The conditioning measures instituted during this session can be outlined as follows. First, the researcher instructed students to begin by sketching the cones before proceeding to depict the plane. This procedural approach was designed to simplify the task's inherent complexity. By breaking down the task into manageable steps, the researcher not only facilitated comprehension for the students but also alleviated potential stress, thereby sustaining their motivation to persist with the task until its completion. Second, in cases where students made errors in positioning the plane intersecting the cone, the researcher provided students with the opportunity to utilize GeoGebra as a tool. This resource allowed students to readily identify and rectify their mistakes. The use of ICT is considered capable of providing several benefits in learning mathematics (Bueno et al., 2023). Consequently, students proceeded to explore alternative plane placements to construct parabolas. Remarkably, even with an initially erroneous plane placement, students ultimately grasped the concept of hyperbolas. The ease with which students acknowledged and corrected their errors was noteworthy.

The challenges faced by students in grasping the concept of conic sections offer a fertile ground for further research. Such investigations can delve deeper into the underlying causes of these difficulties and explore the intricate relationship between abstraction and other variables contributing to these challenges. The findings illuminate some of the obstacles students encounter when mentally visualizing spatial information and conceptualizing the slicing of a cone by a flat plane, resulting in the formation of ellipses, parabolas, and hyperbolas. This underscores the pressing need for extensive research probing the role of spatial thinking in students' abstraction processes. Additionally, difficulties pertaining to drawing and symbolization (representamen) hint at semiotic challenges. Comparable research has proposed mathematical models, highlighting errors in formulating equations (Jupri & Drijvers, 2016). A conspicuous gap between theoretical concepts and empirical data is also discernible in previous studies (Jupri, 2017), characterized by students' struggles to bridge the realms of geometry and algebra. Although spatial and semiotic cognition coexist and interact, there remains a dearth of research examining their intricate interplay. Therefore, there is a compelling need for comprehensive research that meticulously investigates these dual domains.

This research not only serves as a valuable resource for educators in designing abstraction assignments that accommodate students' potential difficulties but also underscores the essential insight that not all students can independently construct knowledge. Consequently, educators are encouraged to develop pedagogical approaches that effectively support students in overcoming these challenges.

CONCLUSION

The research results show that not all students can achieve the desired construction. Twelve out of twenty-five students did not meet the desired construction. One student that interviewed further pass over the all stage of abstraction, namely recognize, building-with and construction. The stages that the student goes through are not always linear and hierarchical, sometimes the student must go through the recognize and build-with stages repeatedly. In abstracting the conic sections, there are several difficulties faced by students, namely difficulties in visualizing the mental image of conical objects, difficulties in spatial thinking and difficulties in using the correct sign. To overcome these difficulties, lecturers can provide a conditioning activity by prompting, probing, the use of tools DGE, so that students can achieve the desired construction.

The utilization of DGE could help students visualizing the mental image of conic section. In the results and discussion, it has been explained how students have recognized old knowledge constructions such as cones and planes (recognize stage). However, drawing a cone cut by a plane that formed an ellipse, parabola, or hyperbola proves to be challenging (building-with and construction stage). This highlights the advantage of using DGE as a tool. Students will find it easier to create new constructions compared to not using it. DGE, like GeoGebra, can be used independently, providing students with the opportunity to construct their own knowledge. Despite its advantages, the use of DGE sometimes presents its own disadvantages in this research. Specifically, due to its instantaneous features, researchers must accurately determine which knowledge students recognize and which they do not. This is precisely what the researchers did in this study: before inviting students to use GeoGebra, the researcher first confirmed the students' existing knowledge. Moreover, students are required to have sufficient knowledge to operate DGE to solve problems effectively.

There is a pressing need for comprehensive research that delves into the intricate interplay between spatial and semiotic cognition. While their coexistence is evident, their interaction remains largely unexplored. Expanding the current study, which focused on one student in an interview, to include a wider sample of students could reveal diverse abstraction processes and unearth new possibilities. This broader research initiative not only fills a critical gap but also equips educators with valuable insights to design abstraction assignments catering to students' potential difficulties. Moreover, it emphasizes the pivotal realization that not all students can independently construct knowledge. Consequently, educators are encouraged to develop pedagogical approaches that effectively support students in overcoming these challenges. Furthermore, considering that the AiC stage framework used in this study only extends to the Construction stage, future researchers are urged to continue investigating how the student consolidation process unfolds beyond this point. This sustained effort could offer a comprehensive understanding of student learning and abstraction, benefiting both academia and pedagogical practices.

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Declarations

Author Contribution	:	MY: Conceptualization, formal analysis, Investigation, Methodology, Visualization, and Writing- original draft. YF: Conceptualization, Methodology, Project administration, Supervision, Validation, Data Analysis, and Writing – review and editing. MTB: Conceptualization, Methodology, Project administration, Supervision, Validation, and Writing – review and editing.
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