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Level of students' proportional reasoning in solving mathematical problems

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Abstract

This study aimed to evaluate the level of proportional reasoning among middle school students in their ability to solve mathematical problems involving proportions. Proportional reasoning is essential for understanding and mastering various mathematical concepts, serving as a fundamental skill for higher-level mathematics. A qualitative case study design was employed, involving 28 eighth-grade students from a school in Bandung, Indonesia. The participants were assessed using a set of proportion-related problems, including numerical comparison, non-proportional (additive), direct proportion, and inverse proportion tasks. The analysis focused on categorizing the students' problem-solving strategies into distinct levels of proportional reasoning, ranging from non-proportional to formal proportional reasoning. Additionally, three students representing high, moderate, and low mathematical performance were selected for in-depth interviews to explore their reasoning processes when addressing proportion problems. Data analysis included administering tests, reviewing students' problem-solving strategies, conducting in-depth interviews, and evaluating their proportional reasoning abilities. The findings revealed that students with high and moderate mathematical performance exhibited proportional reasoning levels ranging from 0 to 3, whereas low-performing students displayed levels ranging from 0 to 2. Moreover, students generally faced difficulties distinguishing between proportional and non-proportional problems. Even when correct answers were provided, many lacked a deep understanding of direct and inverse proportion concepts. The study also discusses several implications for enhancing students' proportional reasoning skills.

Keywords: Middle School Students, Proportional Problem, Proportional Reasoning, Solution Strategies

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A fundamental aim of mathematics education in schools is to enhance students' capabilities in mathematical reasoning (Ministry of Education and Culture, 2022). The importance of developing these reasoning skills is emphasized by research indicating that students frequently engage in imitative reasoning and procedural routines without a comprehensive understanding of the underlying principles (Sukirwan et al., 2018). A crucial aspect of mathematical reasoning is the ability to engage in proportional reasoning (Hjelte et al., 2020). Proportional reasoning is defined as the skill to utilize ratios in contexts involving the comparison of quantities (Doyle et al., 2015; Hilton et al., 2016) and is essential for grasping algebraic concepts as well as facilitating the transition from informal to formal mathematical reasoning

(Doyle et al., 2015). Furthermore, proportional reasoning constitutes a foundational element of elementary mathematics, which is used to analyze proportional situations (Vanluydt et al., 2021). This intricate topic encompasses various relationships and insights that students must comprehend to adeptly apply ratios in a range of mathematical and scientific scenarios (Carney et al., 2016). It necessitates an understanding of the correlation and multiplicative relationships between quantities in these contexts (Misnasanti et al., 2017; Proulx, 2023). Walle et al. (2015) argue that proportional reasoning entails a critical cognitive process for understanding relationships between quantities through multiplication. This viewpoint is supported by Dole et al. (2015), who contend that the capacity to identify the multiplicative relationships between ratios and relevant proportional situations is directly linked to the development of proportional reasoning skills.

The mathematical concept of proportionality is represented through various models, including ratios, the equality of ratios, multiplicative relationships (D'Angela Menduni-Bortoloti & Barbosa, 2017), percentages, and algebraic procedures (Misnasanti et al., 2017). Proportional reasoning is crucial for students' future success in mathematics and scientific disciplines, as it enables them to effectively and flexibly utilize scales and functional relationships to address problems (Carney et al., 2016). Moreover, proportional reasoning is fundamental to comprehending various scientific principles, including density, molarity, speed, acceleration, and force, and it has practical implications in fields such as geography and everyday life (Dole et al., 2015).

Recognizing the importance of mastering proportional reasoning skills, several countries, including Finland, Singapore, the United States, and Türkiye, have prioritized the integration of this skill into their educational curricula (Ayan Civak et al., 2022; Boyer & Levine, 2015). In Indonesia, proportional reasoning plays a central role in the middle school mathematics curriculum, as outlined in the D phase of the Kurikulum Merdeka. In this phase, students are expected to apply mathematical concepts and skills to solve contextual problems, encompassing areas such as numbers, algebra, measurement, geometry, data analysis and probability, and calculus. Consequently, proportional reasoning serves as a foundational basis for understanding and mastering various mathematical concepts.

Proportional Reasoning

According to the literature, three types of problems are frequently employed to assess students' skills in proportional reasoning (Gundogdu & Tunc, 2022). These problem types include missing value problems, numerical comparison problems, and qualitative comparison problems. Missing value problems typically adhere to a standardized format in which three pieces of information are provided, and the objective is to determine the fourth or missing piece of information. Numerical comparison problems present rates or ratios that need to be compared to ascertain whether they are equal, greater, or lesser. In contrast, qualitative comparison problems involve estimation and require comparisons that are not reliant on specific numerical values. For instance, a qualitative comparison problem might ask whether Dana's lemonade will be stronger, weaker, or remain the same compared to yesterday if she uses less juice concentrate and more water today, or if there is insufficient information to make a determination.

Van Dooren classified non-proportional tasks into additive, constant, and linear problems (Gundogdu & Tunc, 2022). In linear problems, the underlying linear function in the problem situation is expressed as $f(x) = ax + b$ with $b \neq 0$. For example, "the locomotive of a train is 12 m long. If there are 4 carriages connected to the locomotive, the train is 52 m long. If eight carriages were connected to the locomotive, how long would the train be?". To solve this problem, students first need to find the length of each carriage. Let us denote the length of each carriage as x meters. So, the total length of the 4 carriages is $4x$ meters. Given that the total length of the train with 4 carriages is 52 meters, we can set up the equation: $52 = 4x + 12$; solving this equation will give the length of each carriage (x)so that students can find the long of the train with 8 carriages connected to the locomotive. Furthermore, Additive problems involve a constant difference between two quantities, requiring the addition of this difference to a third value for a correct approach. To illustrate the problem, “Today, Bert is 2 years old, and Lies is 6 years old. When Bert is 12 years old, how old will Lies be?”. There is a constant difference between Bert's age and Lies’s age; therefore, the relationship between quantities in the situation could be represented as Lies’s age = Bert’s age + 4. Constant problems lack any relationship between the two quantities, where the value of the second quantity remains constant, and the correct answer is explicitly mentioned in the word problem. An example of the constant problem is “Lisa and Linda are planting corn on the same farm. Linda plants 4 rows, and Lisa plants 6 rows. If Linda’s corn is ready to pick in 8 weeks, how many weeks will it take for Lisa’s corn to be ready?”. This situation is constant. Regardless of how many rows were planted, it will still take 8 weeks for the corn to grow This study revealed students' proportional reasoning level in solving mathematical problems, specifically in proportion material. Proportion is a statement that two ratio representatives are equivalent (Lundberg & Kilhamn, 2018; Walle et al., 2015; Wijayanti & Winslow, 2017). There are two proportional relationships, which are direct and inverse proportion. In a direct proportion, two quantities, x and y, remain in a constant ratio relationship (i.e., $x/y = k$). On the other hand, in an inverse proportion, the product of values of two quantities is equal to a constant (i.e., $x * y = k$) (Arican, 2019a).

Numerous researchers have investigated proportional reasoning, particularly focusing on the various levels of this reasoning ability (Gea et al., 2023; Izzatin, 2021; Prayitno et al., 2018, 2019; Sari, 2023). Prayitno et al. (2018, 2019) analyzed students' levels of proportional reasoning, with a primary emphasis on missing value (direct proportion) problems. Their findings indicated that eighth-grade middle school students exhibited non-proportional reasoning at level 0, progressed from level 0 to level 2, and achieved level 4 when addressing missing value problems in proportional reasoning. In contrast, Izzatin (2021) examined students' proportional reasoning levels while solving non-routine problems related to social arithmetic, taking into account their mathematical disposition. The results demonstrated that students with higher mathematical dispositions tended to attain elevated levels of proportional reasoning, and conversely, those with lower dispositions displayed reduced reasoning capabilities. Sari (2023) investigated the proportional reasoning of elementary school students in the context of area conservation, comparing guided and unguided problem-solving approaches. The findings revealed that students' reasoning levels ranged from level 0 to level 2, with no students achieving level 3. Meanwhile, Gea et al. (2023) identified a correlation between students' levels of proportional reasoning and their understanding of fair games in Costa Rica. Their results indicated that as students progressed through grades, their levels of proportional reasoning improved, along with an increasing percentage of students employing effective strategies when addressing fair game problems.

This study also examined students' levels of proportional reasoning, distinguishing itself from prior research by analyzing these levels based on the strategies employed to solve proportion problems, which included numerical comparisons, missing value (both direct and inverse) problems, and non-proportional problems. This approach is informed by earlier studies that identified difficulties students encounter in addressing direct and inverse proportion problems (Bintara & Suhendra, 2021; Irfan et al., 2018, 2019; Wahyuningrum et al., 2019). Ayan and Isiksal-Bostan (2019) and Pelen and Artut (2016) demonstrated that, in general, students often lack a clear understanding of direct and inverse proportions as well as non-proportional problems. Tunc (2020) noted that students frequently resort to inaccurate strategies, particularly proportional strategies, when tackling non-proportional problems. This research makes a significant contribution by providing insights into students' strategies for solving proportion problems, thereby enabling teachers to deliver targeted instruction and interventions aimed at reducing obstacles and potential challenges faced by students. Such challenges serve as indicators for assessing the depth of students' comprehension of specific material. A more detailed explanation of the strategies students employs in solving proportion problems is presented as follow.

Strategies for Solving Proportion Problem

The previous study showed two formal and informal strategies to solve proportional problems (Ben-Chaim et al., 2012; Lamon, 2012). Formal strategies are expressed by cross-multiplication that uses algebra rules and property. In employing this strategy, an equation involving two ratios is solved, with one containing an unknown quantity (Walle et al., 2015). According to (Avcu & Dogan, 2014), cross-multiplication can cause confusion and errors. Moreover, Lamon (2012) states that cross-multiplication does not guarantee proportional reasoning, as an accurate solution can be achieved without recognising structural similarity on both sides of the proportion. Informal strategies include building-up, unit rate, additive, and change factor. Another common informal strategy used is the unit rate strategy, which involves expressing a ratio of a single unit, making it easier to compare or analyse the relationship between the quantities involved (Lamon, 2012). Usually, the additive strategy is imprecise and the most common strategy in solving proportion problems (Ben-Chaim et al., 2012).

An additive strategy involves calculating the difference or sum of two components of a ratio and then dividing the entire quantity by this difference or sum. Additionally, another imprecise strategy occurs when a multiplicative approach, which is appropriate for solving the problem, is incorrectly applied (for example, utilizing a direct proportional formula for an inverse proportion problem) (Ben-Chaim et al., 2012). An illustration of this can be found in the following missing value problem: “Mrs. Mila intends to bake bread. She combines 50 grams of butter with 165 grams of flour. If Mrs. Mila wishes to use 660 grams of flour, how many grams of butter will she require?” Students solving this problem using a formal strategy would typically assign the amount of butter needed as x, form a proportion formula $\frac{660}{165} = \frac{x}{50}$, and subsequently solve the equation using cross-multiplication.

On the other hand, students using the building-up strategy approach the problem by setting a ratio (50 grams of butter to 165 grams of flour) and expanding it to another ratio using addition. They would then solve it using the unit rate strategy, finding the amount of butter needed for 1 gram of flour, which is 0.303 grams of butter. Therefore, for 660 grams of flour, multiplying 660 by 0.303 gives 199.98, which, when rounded, becomes 200 grams of butter. Using the factor of change strategy, if the flour increases by a factor of 4 from the original amount, the butter should also increase by a factor of 4 from the original amount. Thus, the butter needed for 660 grams of flour would be 4 x 50 = 200 grams. An inaccurate additive strategy might involve adding the difference between the butter and flour, 115 grams, to the 660 grams of flour, resulting in 775 grams of butter. Alternatively, if considering the difference between 660 and 165 grams, which is 495 grams, and adding this to 50 grams of butter, the result would be 645 grams of butter.

Level Proportional Reasoning

Following the analysis of the strategies employed by students in solving proportion problems, the investigation into students' levels of proportional reasoning was conducted using the framework established by Langrall and Swafford (2000). The details of this framework are outlined in Table 1.

Table 1. Framework to Assess Students' Proportional Reasoning Level

Level	Strategy Used by Students
L0: Non-proportional Reasoning	▪ Guessing the answer using random numbers, operation or strategy. ▪ Difficulty in recognising the multiplicative relationship from known measurements or using additive or difference strategies. ▪ The method used is still inaccurate and improper with the situation. ▪ Cannot find the relationship between two interrelated measurements.
L1: Informal Reasoning	▪ Used model or manipulation to create a depiction of the existing situation. ▪ Made a qualitative comparison.
L2: Quantitative Reasoning	▪ Used a number line to solve the problem. ▪ Found the unit value of an existing measurement, then used it in solving the overall problem (unit rate). ▪ Identified scalar factor or used scalar factor that can influence changes in existing measurement. Students can also use a table to track changes that occur in existing measurements (factor of change). ▪ Used equivalent fraction. ▪ Build two measurements without using a picture. ▪ Students identified by seeing whether the relationship continues to increase or decrease from the two measurements.
L3: Formal Proportional Reasoning	▪ Created a proportion relationship using a variable, then solved it using the properties of cross-multiplication or equivalent fraction. ▪ Understood the structure of the relationship that exists in each measurement.

METHODS

To ascertain the level of proportional reasoning among middle school students in addressing proportion problems, a qualitative research design utilizing a case study approach was implemented. A case study is employed to investigate a specific issue, event, or phenomenon, facilitating a comprehensive understanding of the subject matter (Crowe et al., 2011; Feagin et al., 2016). In this research, the phenomenon under examination pertains to students' proportional reasoning abilities.

Participants

The study targeted 28 eighth-grade students from one of junior high schools in Bandung. These students were intentionally selected due to their prior exposure to concepts related to equivalent and inverse ratios within their seventh-grade curriculum, as this foundational knowledge was deemed essential for addressing the proportion problems presented in the study. Each student was tasked with solving a series of proportion problems specifically designed to assess their proportional reasoning skills. The researchers conducted a meticulous analysis of the strategies utilized by the students and subsequently selected three individuals for in-depth interviews to further investigate their thought processes and reasoning in solving proportion problems. Following this, the students were grouped according to their levels of proportional reasoning.

The criteria for selecting students for the interviews included: recommendations from mathematics teachers, representation of students with high, moderate, and low mathematics performance, designated as ST, SS, and SR, respectively, and a demonstrated ability for effective communication to ensure that students could articulate their thought processes fluently. This selection of students across varying levels of mathematics performance was intended to determine whether differences exist in the levels of proportional reasoning among these three groups.

Data Collection and Analysis

Data for this research were collected directly by the researchers. The primary data collection instrument consisted of a set of four proportion problems presented in written format. The first problem, adapted from Walle et al. (2015), was designed to assess students' multiplicative thinking abilities. A key characteristic of individuals possessing proportional reasoning skills is their capacity for multiplicative thinking (Dole et al., 2015; Lamon, 2012). The second problem aimed to evaluate students' ability to distinguish between proportional and non-proportional situations, as adapted from Tunc (2020). The third and fourth problems were formulated as missing value tasks.

Prior to administering the instrument to the students, two lecturers specializing in mathematics education were consulted to validate the content of the instrument. Furthermore, the instrument was adapted from previous research that had received validation from mathematics teachers and educators in the field of mathematics education. It was also pilot-tested with both a sixth-grade class and an eighth-grade class to ensure that the problem statements were clear and easily comprehensible. The details of the instruments utilized are presented in Table 2.

Table 2. Proportion Problem to Assess Level of Students' Proportional Reasoning

Number	Problem	Problem Type
1	Five months ago, a farmer measured the height of mango and durian trees; they were 8 inches and 12 inches tall, respectively. The mango tree is 16 inches tall, and the durian tree is 20 inches tall. Over the past five months, which tree's height growth has been the most significant? Explain why.	Numerical Comparison Problem
2	Ridha and Ridho are walking to school at the same speed. Ridho started walking first. When Ridho walked 6 m, Ridha walked 2 m. How far did Ridha walk when Ridho was 12 m away?	Non-proportional (Additive) Problem
3	A bathroom tissue rolling machine from 06:00 until 10:00 AM can produce 120 tissue rolls. After a two-hour break, work continues again until 7.00 PM. How many rolls of tissue does the machine produce in total?	Missing Value Problem (Direct Proportion)

4	Work can be done by four workers in 28 days. The special request came from the customer; the work must be completed more quickly, within 16 days. By adding 2 workers, will the work be completed in the time requested by the customer?	Missing Value Problem (Inverse Proportion)

In the data analysis phase of this research, a systematic approach was employed to ensure rigorous and credible findings. Initially, the researchers identified the various strategies that students utilized when solving proportion problems. To enhance the reliability of this process, two researchers independently reviewed the students' responses, categorizing and documenting the specific strategies each student employed. Following this initial analysis, in-depth interviews were conducted with three selected students representing varying levels of mathematical proficiency: high, moderate, and low. These interviews provided valuable insights into the students' thought processes and reasoning strategies when confronted with proportion problems.

Subsequently, the collected data from both the strategies and interviews were categorized into different levels of proportional reasoning. The classification system used was adapted from the framework proposed by Langrall and Swafford (2000). This involved assigning codes to each level: L0 for non-proportional reasoning, L1 for informal reasoning, L2 for quantitative reasoning, and L3 for formal proportional reasoning. The coding results obtained by the two researchers were then collaboratively reviewed and discussed to ensure consensus, following the methodology suggested by Thomas and Magilvy (2011).

To present the findings comprehensively, the data were summarized descriptively. Finally, based on the analyzed data and the insights gleaned from the interviews, conclusions were drawn to provide a comprehensive understanding of the students' proportional reasoning skills and strategies as shown in Figure 1.

Figure 1. Steps of Data Analysis

The researchers conducted semi-structured interviews with the students to explore their reasoning processes and gain a deeper understanding of their proportional reasoning strategies. During these interviews, students were prompted to discuss their approaches to solving the assigned proportion problems, the rationale behind their selected strategies, their understanding of the relationships between various quantities, and their comprehension of equivalent and inverse ratios.

RESULTS AND DISCUSSION

Students' levels of proportional reasoning are evaluated based on the strategies employed in solving proportion-related problems (see Table 1). The analysis of responses from 28 students revealed that the sole strategy applied to numerical comparison problems was the additive strategy. For non-proportional reasoning problems, students utilized a range of strategies, including an inaccurate additive strategy, a proportional strategy, a number line strategy, and a non-proportional strategy. In the context of numerical comparison problems, students addressed direct proportion problems exclusively using the unit rate strategy. Conversely, for inverse proportion problems, students relied on their intuitive understanding and the proportional equation strategy as presented in Table 3.

Table 3. Frequencies in Type Strategy Solution

Problem	Type of Strategy Solution	Frequency (%)
1	Additive Strategy	28 (100%)
2	Inaccurate Additive Strategy	7 (25%)
	Proportional Strategy	17 (60.7%)
	Number Line Strategy	1 (3.6%)
	Non-proportional Strategy	3 (10.7%)
3	Unit Rate	28 (100%)
4	Intuitive	7 (25%)
	Proportional Strategy	21 (75%)

Following the identification of students' strategies for solving proportion problems, three students were selected for in-depth interviews to further validate their responses and investigate their reasoning processes. These students were intentionally chosen to represent high, moderate, and low levels of mathematical performance. Based on their responses and the outcomes of the interviews, the students' levels of proportional reasoning were assessed. The subsequent analysis yielded comprehensive insights into the students' problem-solving methodologies, illuminating their understanding of proportionality and the strategies they employed to address the given problems.

Students' Proportional Reasoning in Solving Numerical Comparison Problems

In addressing numerical comparison problems, all students exclusively employed the additive strategy. None of the students recognized that the growth of the mango tree over the past five months had increased to twice its initial height, indicating a lack of multiplicative thinking. Figure 2 presents an example of the response from student ST.

Analysis of Figure 2 and the interview findings revealed that the student was unaware of the multiplicative relationship concerning the height of the mango tree. Upon the researcher's guidance, highlighting that 16 and 8 represent a multiplicative relationship of 2 x 8 = 16, ST came to realize that the growth of the mango tree was indeed greater than that of the durian tree. This conclusion stemmed from the observation that the mango tree had doubled its initial height, while the durian tree's growth remained less than double its original height. Consequently, all students, regardless of their proficiency levels—high, moderate, or low—were classified as exhibiting non-proportional reasoning (level 0) when solving numerical comparison problems.

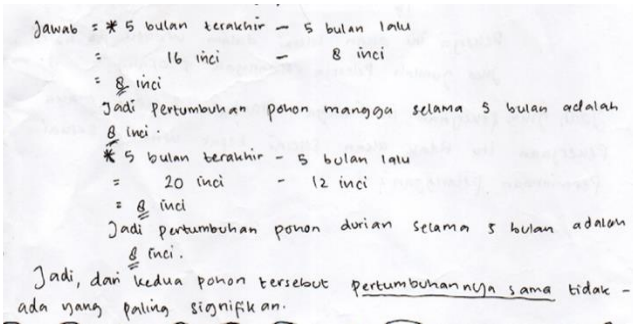
Students answer	**Translation**
	Answer = Last five months – 5 months ago = 16 inches – 8 inches = 8 inches So, the growth of the mango tree in the last five months is 8 inches. = Last 5 months – 5 months ago = 20 inches – 12 inches = 8 inches So, the growth of the durian tree in the last 5 months is 8 inches. Therefore, the growth of both trees is the same, and there is no significant difference.

Figure 2. ST's answer to the numerical comparison problem

The following is an excerpt from an interview with ST

R : Explain the steps you took to solve this problem!
ST : I subtracted the final height of the mango tree from the initial height, ma'am.
R. : Did you think of any other strategy?
ST : No, ma'am. I think my answer is correct.
R : Pay attention to the initial mango height of 8 inches and the final height of 16 inches. Try explaining the relationship between 8 and 16.
ST : 8+8, 16-8

Students' Proportional Reasoning in Solving Non-Proportional (Additive) Problem

Students employed several strategies to address non-proportional reasoning problems, including inaccurate additive strategies, proportional strategies, and non-proportional strategies. The inaccurate additive strategy involved students summing the distances covered by Ridho and Ridha. In contrast, the proportional strategy emerged as the most frequently utilized approach among students in tackling this problem, indicating a belief that the distances between Ridho and Ridha were proportionally related. This misconception highlights students' inability to distinguish between proportional and non-proportional problems. Notably, students with low mathematics performance also approached this problem using a proportional strategy. An example of their response is presented in Figure 3.

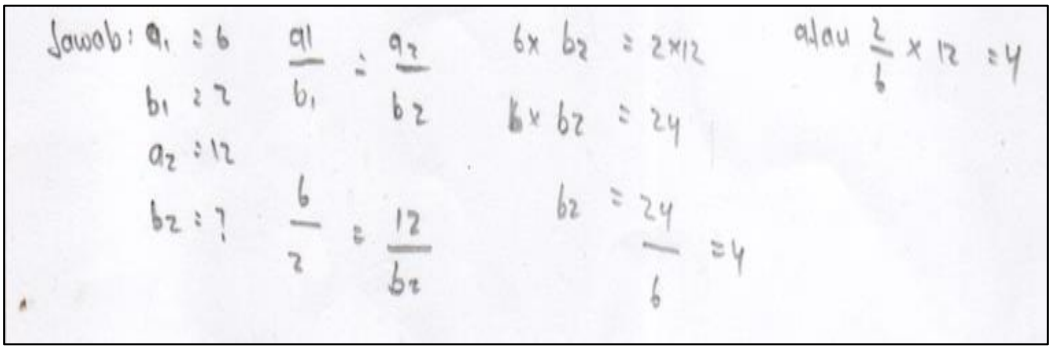


Figure 3. SR's answer to non-proportional reasoning problem

Herewith is the interview excerpt with one of students, namely SR.

R : Can you explain the steps you took to solve this problem?
SR : First, I noted the given information and the question, then I formulated an equation and performed cross-multiplication, ma'am.
R : Why did you create an equation?
SR : Yes, ma'am, because the question provided three known values, so to determine the unknown value, I cross-multiplied.

Based on the interview responses and the student's solution depicted in Figure 3, it is evident that SR perceived the problem in question number 2 as a proportional missing value problem. Consequently, while formulating the solution, SR assumed that the distance ratio between Ridha and Ridho remained constant. Therefore, it can be concluded that SR is categorized at level 0 (non-proportional reasoning), as they failed to identify the additive relationship between the distances of Ridho and Ridha and employed an inaccurate strategy in their solution.

Figure 4 illustrates that SS approached this problem using a number line.

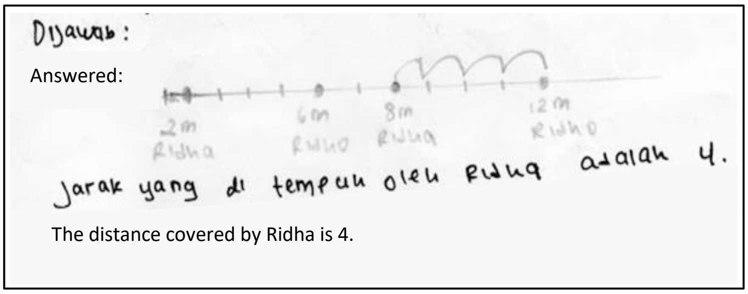


Figure 4. SS's answer to non-proportional reasoning problem

Herewith is the interview excerpt with SS

R : Can you explain the steps you took to solve this problem?
SS : I represented it on a number line, ma'am, and determined that Ridha has walked 4 meters.
R : Please read the question again.
SS : The question asks for Ridha's distance when Ridho is 12 meters away.
R : Now, refer to the number line; when Ridho is at 12 meters, what about Ridha?
SS : When Ridho is at 12 meters, Ridha is at 8 meters, so the distance between them is 4 meters, ma'am.

The results of the interview indicate that SS misunderstood the question, which requested the distance that Ridha had walked, whereas SS responded with the difference in distance between Ridho and Ridha. Consequently, SS's proportional reasoning in solving the problem is classified as level 2 (quantitative reasoning), characterized by the use of a number line to illustrate the problem's context. In contrast to both SR and SS, student ST successfully solved this problem by employing an appropriate strategy (see Figure 5). ST recognized that the problem was non-proportional, and thus, utilized a non-proportional strategy to arrive at the solution. Figure 5 shows the ST's response.

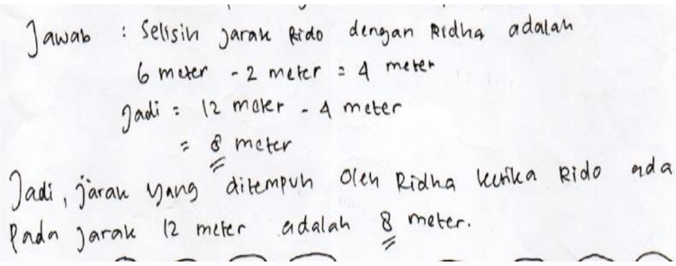
Students answer	Translation
 Jawab : Selisih jarak Rido dengan Ridha adalah 6 meter - 2 meter = 4 meter Jadi = 12 meter - 4 meter = 8 meter Jadi, jarak yang ditempuh oleh Ridha ketika Rido ada pada jarak 12 meter adalah 8 meter.	Answer: The difference in distance between Rido and Ridha is 6 meters - 2 meters = 4 meters. So, = 12 meters - 4 meters = 8 meters. Therefore, the distance Ridha covers when Rido is at a distance of 12 meters is 8 meters.

Figure 5. ST's answer to non-proportional reasoning problem

Interview excerpt with Student ST is as follow.

R : Can you explain the steps you took to solve this problem?
ST : First, I calculated the difference in distance between Ridho and Ridha.
R : Why did you focus on the difference in distance?
ST : Because the problem indicates that Ridho started walking first. When Ridho is 6 meters away and Ridha is 2 meters away, the difference between their distances is 4 meters.

Based on the student's response and the insights gathered from the interview, it is clear that ST comprehends that the distances travelled by Ridha and Ridho are not proportional and employs an appropriate strategy to solve the problem. Consequently, ST is classified at level 3 (formal proportional reasoning).

Students' Proportional Reasoning in Solving Direct Proportion Problems

In addressing the direct proportion problem, all students employed the unit rate strategy. Figure 6 illustrates the responses of three participants.

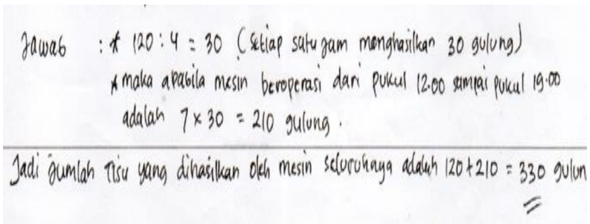
Students answer	Translation
 Jawab : 120 : 4 = 30 (setiap satu jam menghasilkan 30 gulung) maka apabila mesin beroperasi dari pukul 12.00 sampai pukul 19.00 adalah 7 × 30 = 210 gulung. Jadi jumlah tisu yang dihasilkan oleh mesin seluruhnya adalah 120+210 = 330 gulun	Answer: 12: 4 = 30 (Every hour produces 30 rolls) So, if the machine operates from 12:00 to 19:00, it is 7 x 30 = 210 rolls. Therefore, the total number of tissue rolls produced by the machine is 120 + 210 = 330 rolls.

Figure 6. ST's Answer to the Direct Proportion Problem

A noteworthy observation emerged during the interviews; none of the three students recognized that the question pertained to direct proportion. The following is an excerpt from the interview with ST.

R : Can you explain the steps you took to solve this problem?
ST : First, I calculated that the machine produces 120 rolls of tissue in 4 hours, so 120 ÷ 4 = 30. This means it produces 30 rolls per hour. From hour 12 to hour 19, there are 7 hours, resulting in 7 × 30 = 210 rolls. Adding this to the initial 120 rolls gives a total of 120 + 210 = 330 rolls of tissue, ma'am.
R : In your opinion, what is the relationship between the production hours and the number

of rolls of tissue produced?
ST : I do not quite understand.
R : Is it a direct proportion (equivalent) or an inverse proportion?
ST : Um, it is an inverse proportion, ma'am.
R : What is your reasoning?
ST : Sorry, I don't know, ma'am; I forgot.

The interview results indicate that the student does not grasp that the production hours are directly proportional to the number of tissue rolls produced. Furthermore, the student struggles to differentiate between direct and inverse proportion problems, as evidenced by the following examples presented by the researcher.

R : I have two problems. Can you determine whether they represent direct or inverse proportion problems?
1. *The price of 5 candies is IDR 12,000; what is the price of 8 candies?*
2. *Refer to question number 4.*

ST : Both are direct proportion problems, ma'am.
SS : The first one is an inverse proportion question, ma'am. I am uncertain about the second one.
SR : Both are direct proportion problems, ma'am.
R : After studying the material on equivalent and inverse ratios, what do you understand about these concepts?
ST : I recall that equivalent ratios are represented as $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ *, and then we solve using cross-multiplication.*
SS : I believe it is also solved using cross-multiplication.
SR : I forgot the formula, ma'am.

Based on their responses and interview results, ST, SS, and SR can be categorized as level 2 (quantitative reasoning), characterized by their ability to find the unit value of an existing measurement and subsequently use it to solve the overall problem.

Students' Proportional Reasoning in Solving Inverse Proportion Problems

Similarly, the three students were not aware that the problem presented below is one of inverse proportion. Figures 7 and 8 illustrate the answers provided by ST and SS. Additionally, the following is an excerpt from the interview.

R : Explain your steps in solving this problem!
ST : I multiplied the days and workers, then divided them by the days that have been known.
SS : I found the number of works by multiplying the number of days and workers.
S : Why did you multiply it? What is the reason?
ST : That is the formula I remember taught by a teacher about the problem of worker and day.
SS : That is the formula my teacher taught, ma'am.

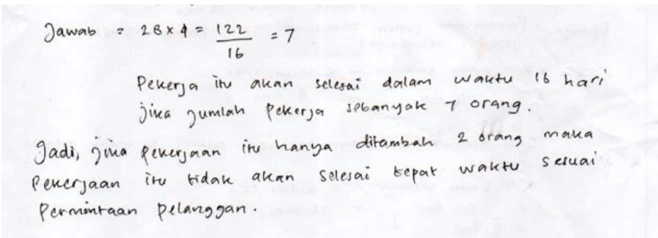
Students answer	Translation
 Jawab : 28 x 4 = 122/16 = 7 Pekerja itu akan selesai dalam waktu 16 hari jika jumlah pekerja sebanyak 7 orang. Jadi, jika pekerjaan itu hanya ditambah 2 orang maka pekerjaan itu tidak akan selesai tepat waktu sesuai permintaan pelanggan.	Answer: $28 \times 4 = \frac{122}{16} = 7$ The worker will finish within 16 days if the number of workers is seven people. So, if only 2 more workers are added, the job will not be completed on time per the customer's request.

Figure 7. ST's answer to the inverse proportion problem

ST and SS successfully solved the problem by recalling the formulas taught by their instructors. However, they lacked an understanding of the relationship between the two quantities presented in the problem. This was evident when they were presented with another scenario: "Five friends plan to purchase a gift for their friend's birthday, with each individual contributing Rp. 3,000. If four additional friends decide to join, how much will each person contribute?" Both ST and SS identified this as an equivalent ratio problem and swiftly resolved it using cross-multiplication.

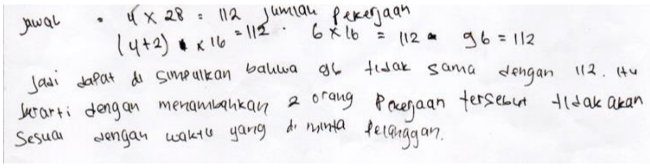
Students answer	Translation
 Jawab : 4 x 28 = 112 jumlah pekerjaan (4+2) x 16 = 112 . 6 x 16 = 112 96 = 112 Jadi dapat di simpulkan bahwa 96 tidak sama dengan 112. itu berarti dengan menambahkan 2 orang pekerjaan tersebut tidak akan sesuai dengan waktu yang di minta pelanggan.	Answer: 4 x 28 = 112 total workload (4+2) x 16 = 112 6 x 16 = 112 96 = 112 So, it can be concluded that 96 is not equal to 112. This means that by adding 2 more people, the job will not meet the customer's requested time.

Figure 8. SS's answer to inverse proportion problem

Although ST and SS successfully applied the correct strategy to solve the problem, they failed to recognize the underlying relationship between the quantities involved. Consequently, both students are classified at level 0, indicating non-proportional reasoning. In contrast, SR demonstrated a different understanding, as illustrated in Figure 9, reflecting a more comprehensive grasp of the problem's context and the proportional relationships within it.

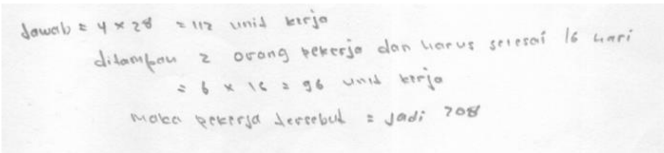
Students answer	Translation
 Jawab = 4 x 28 = 112 unit kerja ditambah 2 orang pekerja dan harus selesai 16 hari = 6 x 16 = 96 unit kerja Maka pekerja tersebut = jadi 208	Answer: 4 x 28 = 112 units of work. Adding 2 more workers and needing to finish in 16 days = 6 x 16 = 96 units of work. So, the total work becomes 208.

Figure 9. SR's answer to the inverse proportion problem

The interview with SR reveals a lack of understanding of the correct strategy to solve the problem. Although SR claimed to understand the question, they admitted to forgetting the formula and resorting to guessing the answer.

R : Can you explain the strategy you used to answer this problem?
SR : I forgot the formula, ma'am, so I randomly guessed it.
R : But do you understand what this problem asks?
SR : I understand it, ma'am. I just forgot the formula.

Based on the analysis of SR's written response in Figure 9 and the interview, SR is classified at level 0, indicating non-proportional reasoning. This conclusion stems from SR's inability to identify the relationship between quantities and the use of an incorrect strategy. The following section outlines the proportional reasoning levels for all three respondents, based on their written answers and interview insights.

Table 4. Profile of proportional reasoning level respondent

Mathematical Skill	Respondent	Level of Proportional Reasoning			
		Problem 1	Problem 2	Problem 3	Problem 4
High	ST	L0	L3	L2	L0
Moderate	SS	L0	L2	L2	L0
Low	SR	L0	L0	L2	L0

Table 4 highlights that there is no significant difference in the proportional reasoning levels of students with high, moderate, or low mathematical performance when solving numerical comparison problems and missing value problems involving direct and inverse proportions. However, a notable difference emerges when students tackle non-proportional problems. Students with high mathematical performance reached level 3, formal proportional reasoning, indicating their ability to understand the structural relationship between two quantities. Those with moderate performance were categorized at level 2, quantitative reasoning, where they relied on visual aids such as number lines to solve the problem. Finally, students with lower mathematical abilities were placed at level 0, non-proportional reasoning, as they could not identify the correct relationship between quantities. A more detailed analysis of the students' answers and interview results follows to provide further insights into their reasoning levels.

Discussion

This research demonstrates that students display a range of proportional reasoning abilities when addressing proportion-related problems, spanning from level 0 (non-proportional reasoning) to level 3 (formal proportional reasoning). Students at level 0, indicative of non-proportional reasoning, encounter significant difficulties with numerical comparison, non-proportional, and inverse proportion problems. They are unable to recognize multiplicative relationships between quantities, often confuse proportional with non-proportional problems, and may resort to intuitive or incorrect strategies when faced with inverse proportion tasks. The ability to think multiplicatively is a key attribute of students who possess proportional reasoning skills (Dole et al., 2015). In mathematics education, particularly at the middle school level and beyond, proficiency in various aspects of multiplicative thinking is essential. Students who lack this capability tend to struggle with understanding and applying multiplicative concepts and operations (Jitendra et al., 2023). This deficiency can hinder their ability to reason about multiplicative relationships and solve more complex mathematical problems effectively (Venkat & Mathews, 2018).

In inverse proportion problems, students with high and moderate mathematical abilities can apply the correct strategies. However, when probed further, these students do not fully comprehend the relationship between the two quantities involved. When presented with inverse proportion problems in different contexts, they incorrectly apply strategies, often treating them as equivalent ratio problems. This suggests that their approach to problem-solving is highly dependent on the context and limited to the mechanical use of the cross-multiplication formula. Their recall centers on associating three-number problems with cross-multiplication, while problems involving workers and time trigger the application of specific formulas taught by their teachers. Students with lower abilities, on the other hand, tend to rely on intuitive strategies when solving inverse proportion problems. These findings are consistent with previous research (Parameswari et al., 2023), which highlights students' challenges in distinguishing direct from inverse proportion problems, as shown in earlier studies (Arican, 2019b; Ayan & Isiksal-Bostan, 2019; Irfan et al., 2018, 2019).

According to Piaget's theory of cognitive development, proportional reasoning is a key indicator of the formal operational stage, where abstract thinking becomes possible (Vanluydt et al., 2020; 2021). Students who have not reached this stage struggle with tasks requiring advanced abstract reasoning, such as differentiating between direct and inverse proportions. Notably, several interview results in this study show that many middle school students have not yet reached the formal operational stage (Putra, 2014; Rohaeti et al., 2019).

Furthermore, many students in this study incorrectly apply proportional strategies to non-proportional problems. As illustrated in Figure 3, students fail to recognize that the relationship between variables in certain problems is additive rather than multiplicative. This leads them to use the cross-multiplication method inappropriately, reflecting their difficulty in distinguishing between proportional and non-proportional problems. This challenge is consistent with findings from previous studies (Arican, 2019a; Artut & Pelen, 2015; Atabaş & Öner, 2017; Tunc, 2020; Van Dooren et al., 2005).

Notably, level 1 or informal reasoning did not emerge in this study. Informal reasoning typically involves students making qualitative comparisons or using visual representations to relate quantities. Level 2, quantitative reasoning, was observed in the context of solving missing value problems, specifically in direct proportion scenarios. A key feature of this level is the use of the unit rate strategy to solve such problems. Interestingly, in contrast to the findings of Tunc's research, where students predominantly applied cross-multiplication to tackle missing value problems, all participants in this study relied on the unit rate strategy. However, a notable aspect emerged when students were presented with a direct proportion problem framed differently, containing only three known numbers. In these cases, students swiftly reverted to using cross-multiplication, indicating that their problem-solving strategies are heavily influenced by the context of the question.

According to Izsák and Jacobson (2017), one major challenge in understanding proportional relationships stems from traditional teaching methods, which emphasize memorizing rules and performing routine calculations. The heavy reliance on cross-multiplication and rote calculation strategies, as highlighted by Arican (2018, 2019b), is evident in how students approach proportion problems. While cross-multiplication is an efficient tool for solving comparison problems, it can hinder both students' and teachers' deeper understanding of the multiplicative relationships between quantities (Lamon, 2012). This aligns with claims made by Avcu and Dogan (2014), who argue that the cross-multiplication strategy lacks a physical or intuitive referent and therefore carries little meaning for students. Given this, it is essential that cross-multiplication is not introduced until students have ample experience with intuitive and conceptual strategies, such as the factor-of-change, build-up, or unit rate approaches. These strategies provide a more meaningful foundation for understanding proportional relationships.

Level 3, or formal proportional reasoning, is the highest level attained by students in this study, although it was not widely exhibited. Walle et al. (2015) and Lamon (2012) define several key characteristics of a proportional thinker: (1) understanding covariance, which means recognizing how two quantities vary together and how changes in one correspond to changes in the other; (2) distinguishing proportional from non-proportional relationships in real-world contexts; (3) developing strategies for solving proportion problems or comparing ratios, usually based on formal strategies rather than memorized algorithms; (4) viewing ratios as independent entities that express distinct relationships between quantities; and (5) comprehending both direct and inverse proportional relationships.

The findings of this research suggest several important implications for improving students' proportional reasoning skills. First, it is essential to emphasize the development of multiplicative understanding, as this forms the foundation of proportional reasoning. Dole et al. (2015) highlight that grasping the multiplicative structures inherent in proportional situations is critical. This can be facilitated through targeted instructional materials and activities that focus on the relationships between quantities. Teachers can employ various strategies to nurture multiplicative thinking, including encouraging critical thinking, promoting classroom discussions, and using multiple representations (e.g., diagrams, ratio tables, or models) to illustrate multiplicative relationships (Jitendra et al., 2023).

Secondly, mathematics textbooks should place greater emphasis on distinguishing between proportional and non-proportional problems. Providing clear examples and distinct solving strategies for each type of problem can help students recognize and approach these problems more effectively. This is consistent with the recommendations of Burgos & Godino (2020), who advocate for the use of diverse contexts that engage students in multiplicative reasoning and draw upon their prior knowledge.

Finally, building on the research of Vanluydt et al. (2020), it is crucial for teachers to possess a deep understanding of proportional reasoning. This includes the ability to identify proportional situations, facilitate student development in this area, understand the mathematical structures in proportional contexts, and grasp the constant relationships between quantities expressed through ratios. Additionally, educators should recognize that proportional reasoning involves multiplicative comparisons and employ various tools, such as ratio tables and double number lines, to support students' reasoning. By cultivating these competencies, teachers can effectively guide students through the complexities of proportional reasoning, leading to a deeper conceptual understanding of mathematical relationships.

CONCLUSION

The levels of proportional reasoning among students when solving proportion problems vary from level 0 to level 3. However, it is notable that students with high, moderate, or low skills in solving numerical comparison, direct, and inverse proportion problems do not show significant differences in their proportional reasoning levels. Overall, students' proportional reasoning, particularly in numerical comparison, non-proportional, and inverse proportion problems, tends to remain at level 0, reflecting a non-proportional reasoning approach. At this level, students struggle to recognize multiplicative relationships between quantities and face difficulties distinguishing between proportional and non-proportional problems. They often rely on intuition or employ incorrect strategies to address such problems.

This trend suggests a deeper issue that warrants further research. Specifically, future studies should explore why students' proportional reasoning abilities are not sufficiently developed, particularly when dealing with proportion topics. Additionally, analyzing the types of proportional reasoning tasks presented in textbooks used in schools could offer valuable insights into the factors that influence students' understanding and performance in this area of mathematics. Understanding these factors is crucial for enhancing the teaching and learning of proportional reasoning in classrooms, thereby improving students' ability to navigate complex mathematical relationships.

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