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Investigating the effect of manipulatives on mathematics achievement: The role of concrete and virtual manipulatives for diverse achievement level groups

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Abstract

This study investigates the impact of integrating both concrete and virtual manipulatives on the mathematics achievement of fifth-grade students across different achievement levels (low, average, and high). Utilizing a quasi-experimental design with pre- and post-tests, a convenience sample of fifth-grade students was randomly assigned to either a control or experimental group. The data collection instruments, consisting of separate pre- and post-tests on the same mathematical concepts, underwent thorough validity and reliability testing. Initial assessments demonstrated that the achievement levels between the control and experimental groups were comparable prior to the intervention. The experimental group received instruction that incorporated both concrete and virtual manipulatives, whereas the control group followed traditional teaching methods. Following a 12-week intervention period, a post-test was administered. The data were analyzed using parametric paired-sample t-tests and one-way ANCOVA, ensuring that all underlying assumptions were satisfied. The findings revealed significant improvements in post-test scores among students in the experimental group, regardless of their initial achievement levels. Although low-achieving students in the control group also showed progress, their gains were less substantial compared to those in the experimental group. This study highlights the potential benefits of incorporating both concrete and virtual manipulatives in fifth-grade mathematics instruction to enhance academic achievement.

Keywords: Combination of Concrete and Virtual Manipulatives, Different Achievement Level Groups, Fifth Graders, Mathematics Achievement

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Mathematics frequently necessitates that students comprehend abstract concepts and engage in complex problem-solving (Nurjannah & Kusnandi, 2021). Research conducted by Sugiarti and Retnawati (2019) and Sulistiowati et al. (2019) indicates that numerous students encounter challenges in understanding abstract algebra, geometry, and measurement concepts, particularly at the primary and junior high school levels. These difficulties adversely affect their academic performance in mathematics (Siller & Ahmad, 2024). In Pakistan, the situation is especially concerning at the primary education level in public schools, with grade five students in Punjab province facing significant challenges. The national curriculum for grade five mathematics encompasses three principal areas: numbers (including arithmetic operations, factors and multiples, fractions, decimals, and percentages), measurement and geometry, and information handling. Proficiency in these areas is critical for advancing to more complex mathematical concepts in subsequent grades (Bajpai & Pandey, 2024). Large-scale assessment (LSA) data reveal that, on average, only 41% of students in Punjab are able to solve problems related to number units as outlined in the curriculum. Moreover, only 37.8% of students are proficient in solving measurement-related problems, 39.2% in geometry-related problems, and 43.2% in information handling questions. These results, derived from a sample of 4,478 students in Punjab, underscore the persistent deficiencies in mathematical problem-solving skills, which extend into higher educational levels (LSA, 2021).

Punjab, which represents over half of Pakistan's total population, plays a critical role in national educational statistics (Dhindsa, 2020). National and international assessments have repeatedly highlighted the suboptimal performance of students in Punjab, especially in mathematics (PASL, 2020; Asian Development Bank, 2023). For instance, the Learning and Educational Achievements in Punjab Schools (LEAPS) report indicated that by the conclusion of third grade, only 38% of students were able to solve a basic two-digit subtraction problem, such as "238-129" (Andrabi et al., 2008). Furthermore, the 2023 Annual Status of Education Report (ASER) for Pakistan revealed that 39% of students enrolled in public schools were unable to solve basic two-digit division problems (ASER, 2024). The Provisional Assessment of Student Learning (PASL) report for 2018-2019 documented that primary-level students are performing significantly below average, adversely affecting their mathematical understanding and performance in higher grades (PASL, 2020). These observations are corroborated by a nationwide study conducted by Bhutta and Rizvi (2022). The underlying causes of this inadequate performance are complex and include various classroom and school-related factors. Interactive classroom environments and effective pedagogical approaches are identified as critical determinants influencing student performance (PASL, 2020; Bhutta & Rizvi, 2022).

In Pakistan, the instruction of mathematics predominantly adheres to a conventional, deductive methodology. Educators generally commence lessons by presenting mathematical formulas, which are written on the board for students to memorize and apply in subsequent practice exercises. This approach tends to render students' passive recipients of information, with their engagement largely limited to responding to direct inquiries from the teacher and often minimal interaction with peers (Iqbal et al., 2020; Siller & Ahmad, 2024). Furthermore, mathematics education in Pakistan has historically emphasized knowledge transmission rather than conceptual understanding, with direct instruction being perceived as the most efficacious pedagogical strategy (ASER, 2024). A typical mathematics lesson involves the introduction of a new concept through illustrative examples, followed by a step-by-step demonstration by the teacher. Students are then assigned notes and textbook problems to solve, while the teacher monitors their progress (Mirza & Iqbal, 2014). The curriculum places substantial reliance on official textbooks, which present knowledge as fixed and to be acquired through rote memorization and repetitive exercises. In this framework, textbooks significantly influence the dynamics between teachers and students. The teaching methodology reinforces the teacher's authority as the primary source of mathematical knowledge, leading to a rigid, authoritative, and often impersonal classroom environment (Iqbal et al., 2020).

Internationally, the National Council of Teachers of Mathematics (NCTM) endorses the incorporation of multiple representations—such as pictorial, manipulative, graphical, and numerical—within mathematics instruction. This multifaceted approach aids students in exploring abstract mathematical concepts and enhances their mathematical achievement (Mainali, 2021). Utilizing various interconnected representations enables students to establish connections between different forms of the same concept, thereby facilitating the transition from concrete to abstract understanding (NCTM, 2014).

Empirical research by Dinsmoor (2022) and Crowe (2022) supports the effectiveness of representational models in engaging students and improving their mathematical learning outcomes. Specifically, the use of manipulatives has been recognized as one of the most effective strategies for teaching and learning mathematics at the primary education level (Nash, 2023; Lange, 2021; Milton et al., 2023). In the Pakistani context, both the Federal and Provincial Curriculum Wings and textbook publishing authorities advocate for an interactive, student-centered pedagogical approach. They emphasize the importance of involving students in the learning process through material-supported teaching, commonly known as manipulative-assisted instruction in mathematics (National Curriculum Council – Pakistan, 2020).

Manipulatives are pedagogical tools centered around student engagement that facilitate active problem-solving and inquiry through the hands-on exploration of physical objects designed to represent mathematical concepts (Dinsmoor, 2022). This method is consistent with constructivist learning theory, which posits that students construct their own knowledge through direct interaction and exploration (Ernest et al., 2016). The use of manipulatives contributes to the development of a robust foundational understanding and a deeper comprehension of mathematical principles (Dinsmoor, 2022; Crowe, 2022). For example, manipulatives are often employed to teach arithmetic concepts; a common practice involves using physical objects, such as adding two toy parrots to a group of four, thereby demonstrating that 4 + 2 = 6 through a tangible activity (Foulkes et al., 2023). Educators in prekindergarten through middle school frequently utilize both concrete models and digital manipulatives as instructional aids. These tools support students' understanding and enhance their mathematical achievement across a range of domains, including number operations, geometry, algebra, measurement, data analysis, and probability (Johnson, 2022).

Mathematical manipulatives, encompassing both concrete and virtual forms, represent a well-established instructional strategy in mathematics education, with substantial research support (Shuxratovna, 2024; Bone et al., 2023; Ukdem & Çetin, 2022; Sari & Aydoğdu, 2020; Kabel et al., 2021; Surynková, 2023). Empirical evidence demonstrates that the integration of manipulatives in mathematics instruction significantly enhances student achievement (Wilkie & Hopkins, 2024; Shen, 2023; Back, 2019; Tjandra, 2023). While manipulatives are frequently employed in primary education, their efficacy has been substantiated across all educational levels (Julie, 2021). Manipulatives are generally categorized into two primary types: concrete and virtual (Ye & Feng, 2019). Concrete manipulatives consist of physical materials or tools, such as pattern blocks, algebra tiles, fraction strips, and geoboards (Stigberg et al., 2022). These tools are cost-effective, do not require an external power source, and are straightforward to use (Gilligan-Lee et al., 2023; Jones & Tiller, 2017). In contrast, the past two decades have witnessed an expansion in the range of manipulatives to include digital or virtual alternatives (Shurr et al., 2021; Root et al., 2021). Virtual manipulatives replicate physical teaching aids and can be accessed through networked platforms or software (Zengin, 2023; Guan et al., 2020). Examples of virtual manipulatives include base-ten blocks, Cuisenaire rods, and fraction circles available on platforms such as MathConceptua, Illuminations, and the National Library of Virtual Manipulatives (Moyer-Packenham & Bolyard, 2016).

The use of concrete manipulatives in mathematics instruction allows students to physically interact with objects that symbolize mathematical concepts, thereby enhancing conceptual understanding and improving mathematical achievement (Prosser & Bismarck, 2023; Milton et al., 2023). Recent intervention studies indicate that students instructed with concrete manipulatives demonstrate superior performance compared to those who receive instruction without these tools (Lange, 2021; Simon, 2022; Ukdem & Çetin, 2022; Bornaa et al., 2023; Muhammad et al., 2023). Additionally, research comparing the impact of concrete versus virtual manipulatives on academic achievement in mathematics has identified notable differences between the two approaches (Back, 2019; Kabel et al., 2021; Lange, 2021; Hawes et al., 2022; Tjandra, 2023; Shafiq et al., 2023). Although concrete manipulatives are effective in enhancing students' mathematical understanding and performance, they present certain challenges, such as difficulties in mobility, substantial classroom space requirements, and the considerable time needed for setup and teardown (Satsangi & Miller, 2017; Salifu et al., 2022). Virtual manipulatives offer a solution to these limitations by providing a more adaptable and efficient alternative for mathematics instruction (Satsangi & Bouck, 2015).

Virtual manipulatives are interactive digital representations of mathematical objects that can be displayed on electronic devices such as computers, tablets, and smartphones. These manipulatives are accessible through applications, software programs, or online platforms (Bouck et al., 2018). Many virtual manipulatives offer customizable configurations, which provide additional support and guidance for students who encounter difficulties with mathematical concepts (Long et al., 2023; Yakubova et al., 2024; Haji Ismail et al., 2023; Bouck et al., 2023). These features promote student autonomy and may include hints, extensions, and real-time feedback, thereby enhancing the learning experience (Kabel et al., 2021; Liu et al., 2024). The primary objective of virtual manipulatives is to digitally represent mathematical concepts and allow students to interact with dynamic objects on the screen (Abdul-Karim et al., 2023; Yakubova et al., 2024). Interactive visual models facilitate additional practice opportunities on virtual platforms, provide scaffolded mathematical content, and ultimately contribute to improved student achievement in mathematics (Satsangi et al., 2021; Ukdem & Çetin, 2022; Serin, 2023; Akpan et al., 2023; Liu et al., 2024).

Extensive research highlights the benefits of utilizing virtual manipulatives in mathematics education (Satsangi et al., 2018; Bouck et al., 2019; Peltier et al., 2020; Bouck et al., 2020; Shin et al., 2021; Park & Bouck, 2022; Shin et al., 2023; Satsangi & Raines, 2023). For instance, Lee and Chen (2015) conducted a study with tenth-grade students and found that virtual manipulatives were as effective as concrete manipulatives in improving mathematical achievement, while also enhancing the overall enjoyment of the learning process. Table 1 provides a summary of various virtual manipulatives, including both web-based and application-based tools, and their respective features.

Table 1. Summary of the features of virtual manipulatives

Name and URL	Web App	iPad App	Visual Model	Corrective Feedback	Audio	Video Tutorial
Conceptual Math http://www.conceptuamath.com	Yes	Yes	Multiple	Yes	Yes	Yes
Illuminations http://illuminations.nctm.org/Games-Puzzles.aspx	Yes	Yes	Multiple	Yes	No	No
NLVM http://nlvm.usu.edu/en/nav/vlibrary.html	Yes	No	Multiple	Yes	No	No
Virtual Manipulatives http://ABCya.com	No	Yes	Multiple	No	No	No

Lange (2021) indicates that employing both concrete manipulatives (such as blocks or counters) and digital manipulatives (including interactive applications or online tools) can enhance students' comprehension of mathematical concepts. Over the past decade, scholars have investigated the comparative effectiveness of these two approaches, particularly for instructing young learners in mathematics. While findings have been varied, the majority of studies suggest that digital manipulatives can be as effective as, if not superior to, concrete manipulatives in terms of improving students' mathematical achievement and understanding (Pavlou et al., 2024; Mazo, 2024; Guan et al., 2024; Shen, 2023; Ng & Ye, 2022; Byrne et al., 2023). Recent research has begun to explore the integration of both types of manipulatives, aiming to leverage the strengths of each to enhance instructional efficacy (Wang & Tseng, 2018). Studies by Wang and Tseng (2018) and Zacharias and de Jong (2014) have demonstrated that a blended approach, whether starting with concrete or digital manipulatives, yields superior outcomes in terms of test scores and conceptual understanding compared to using either type alone. In our study, we opted to begin with physical manipulatives before introducing digital ones to the experimental group.

After reviewing the current literature, it is evident that there is no conclusive evidence favoring either concrete or virtual manipulative-assisted methodologies in mathematics instruction (Mazo, 2024; Guan et al., 2024; Shen, 2023; Coles & Sinclair, 2019; O'Meara et al., 2020; Gravito et al., 2023; Jablonski & Matthias, 2023). Some studies advocate for a combined approach to leverage the benefits of both types of manipulatives (Wang & Tseng, 2018). Prior research has predominantly explored this integration in subjects such as science and physics across various age groups (Zacharias & de Jong, 2014; Zacharias & Olympiou, 2011; Toth et al., 2009). Nevertheless, there remains a significant gap in understanding how students with varying levels of achievement might benefit from a combination of concrete and virtual manipulatives in fifth-grade mathematics education. As primary-level mathematics instruction continues to advance, there is an increasing interest in exploring innovative methods such as the use of manipulatives. This approach is supported by organizations such as the National Curriculum Council of Pakistan and the National Council of Teachers of Mathematics (NCTM). Despite the widespread use of manipulatives in primary math classrooms globally, their implementation in countries like Pakistan is relatively recent. Therefore, further research is required to comprehend the effects and implications of combining concrete and virtual manipulatives in educational settings.

This paper addresses a significant research gap by presenting findings from a teaching experiment conducted in fifth-grade classrooms. Recent global debates have focused on the relative effectiveness of concrete versus digital manipulatives for teaching mathematics across various age groups. This experimental study involved students with diverse achievement levels, who engaged with a combination of concrete and virtual manipulatives to develop proficiency in a range of mathematical concepts, including whole numbers, fractions, decimals, percentages, unitary methods, geometry, and the calculation of perimeters and areas. Given the scarcity of research on how combining concrete and virtual manipulatives impacts students' mathematical achievement at the primary level, this study provides valuable insights into the field of mathematics education. The primary objective of the experiment was to assess the impact of integrating both concrete and virtual manipulatives on the mathematical achievement of fifth-grade students across different achievement levels. This study aims to determine whether this blended manipulative-assisted instructional approach is equally effective for students with varying levels of mathematical proficiency.

The subsequent section will outline the development of the research questions, which have been shaped by a comprehensive review of existing literature and the identification of existing research needs and gaps.

Research Questions

Several research have demonstrated that the deliberate integration of both concrete and virtual manipulatives in mathematics instruction can significantly enhance student engagement and academic performance (Zacharias & Olympiou, 2011; Mazo, 2024; Guan et al., 2024; Shen, 2023; Ng & Ye, 2022; Byrne et al., 2023; Gravito et al., 2023; Jablonski & Matthias, 2023; Shuxratovna, 2024; Bone et al., 2023; Ukdem & Çetin, 2022; Sari & Aydoğdu, 2020; Kabel et al., 2021; Surynková, 2023; Tjandra, 2023; Muhammad et al., 2023; Shafiq et al., 2023). Building on this evidence, the current study seeks to investigate whether a combined approach of concrete and virtual manipulatives can improve mathematics achievement across various achievement levels, specifically for fifth-grade students, when compared to traditional teacher-centered methods. This leads to the formulation of the following research questions (RQs):

RQ 1: How do mathematics manipulatives impact fifth graders' mathematics performance across different achievement levels?

RQ 2: Is there a significant difference in mathematics achievement between low achievers who receive manipulative-based instruction and those who receive traditional instruction?

METHODS

Study Design

This study aimed to investigate the combined effects of concrete and virtual manipulatives on fifth graders' mathematics achievement using a quasi-experimental quantitative research design with a pre-posttest approach. The study involved two groups of students: an experimental group and a control group (Mills & Gay, 2018). The experimental group utilized a combination of concrete and virtual manipulatives in their mathematics instruction, while the control group received traditional instruction. The design allowed for a comparative analysis of the impact of the manipulative-based instructional model versus conventional teaching methods on student performance.

Sample

The study focused on fifth graders at an elementary school in Pakistan, addressing a significant issue of poor mathematics performance observed in this demographic. As detailed in the introduction, this problem underscores the importance of fifth grade as a pivotal stage in mathematics education. Mastery of fundamental concepts such as whole numbers, fractions, decimals, and numerical expressions is crucial, as these skills form the basis for more advanced mathematical topics like algebra and geometry encountered in middle school (National Curriculum Council – Pakistan, 2020; Bajpai & Pandey, 2024). Research indicates that early numeracy skills and counting competencies established during preschool and elementary education are strong predictors of future mathematics achievement, including performance in fifth grade and beyond (Nguyen et al., 2016). Additionally, a solid mathematical foundation by the end of fifth grade is essential for fostering students' confidence, interest, and positive attitudes towards mathematics, which are critical for their ongoing academic success and for keeping pace with their peers (Siller & Ahmad, 2024; Bajpai & Pandey, 2024).

The study engaged a total of 87 fifth-grade participants, divided into 45 students in the experimental group and 42 in the control group. This sample size was chosen based on the recommendations by Mills and Gay (2018), which advocate for a minimum of 30 participants per group to ensure the validity of statistical analyses. This approach is consistent with recent STEM education research, which often employs similar sample sizes (Yaduvanshi & Singh, 2019). Thus, selecting 87 students, with 45 in the experimental group and 42 in the control group, was intended to guarantee the statistical robustness of the findings.

Participants were randomly selected using readily accessible sampling, which was guided by factors such as ease of access, willingness to participate, and availability during the study period (Arrogante, 2022). All participants completed pre- and post-tests to measure their mathematical achievement. To ensure comparability between groups, pre-test scores were analyzed to confirm that the experimental and control groups began at similar performance levels.

Following this, students were stratified into low-, average-, and high-achieving groups based on their pre-test scores. In the experimental group, there were 24 low achievers (N=24), 10 average achievers (N=10), and 11 high achievers (N=11). The control group comprised 22 low achievers (N=22), 13 average achievers (N=13), and 7 high achievers (N=7). This stratification aimed to evaluate the effectiveness of the manipulative-assisted instruction across different achievement levels.

The categorization criteria were as follows: low achievers scored between 10-25 out of 64, average achievers scored between 26-40, and high achievers scored between 41-64. These criteria were adapted from recent studies by Ghodbane and Achachi (2019) and Yaduvanshi and Singh (2019). By grouping students into these achievement levels, the study aimed to assess whether the manipulative-assisted intervention was effective across all levels or if different strategies were needed for students with varying levels of achievement. This approach sought to determine if a single intervention strategy was equally effective for all students or if tailored strategies were required for low, average, and high achievers.

Test Instruments

To address the research questions, two mathematics achievement tests, namely Pre-MAT and Post-MAT, were employed. Each test consisted of 100 items, which were meticulously developed based on key concepts from the fifth-grade mathematics curriculum provided by the Punjab Curriculum and Textbook Board, Pakistan. The development of these tests involved rigorous validation processes to ensure their reliability and relevance.

Four experts in instrument development evaluated the technical aspects of construct validity for both tests, while four mathematics content specialists verified their face and content validity. This dual-layered validation process aimed to ensure that the tests accurately and comprehensively assessed the intended mathematical concepts.

The Pre-MAT and Post-MAT tests were designed to measure students' proficiency in various areas of the fifth-grade mathematics curriculum. These areas included:

1. Whole Numbers: Concepts such as number expansion, multiplication, division, and patterns.
2. Fractions: Operations including addition, subtraction, multiplication, and division of fractions.
3. Decimals and Percentages: Skills related to ordering decimals, performing arithmetic operations (addition, subtraction, multiplication, division), and converting between fractions, decimals, and percentages.
4. Unitary Method: Application of the unitary method to solve real-life problems involving addition, subtraction, multiplication, and division.
5. Geometry: Fundamental concepts including angles, triangles, quadrilaterals, and symmetry.
6. Perimeters and Area: Calculation of perimeters and areas for geometric shapes such as squares, rectangles, and parallelograms.

These tests were administered to assess students' mathematical achievements at the beginning and end of the intervention period, providing a basis for evaluating the impact of the concrete and virtual manipulatives on their learning outcomes.

The Pre-MAT and Post-MAT tests were designed with differing question items in terms of face and content validity, while maintaining consistent assessment of the core mathematical concepts. To ensure the reliability and effectiveness of these tests, a pilot study was conducted with a separate group of fifth-grade students.

The pilot study involved 243 students from public schools with similar demographic and environmental backgrounds. This study was carried out in two phases: the winter of 2022/2023 for the pretest and the summer of 2023 for the posttest. Importantly, these pilot study participants were distinct from those eventually selected for the main experiment. This separation was intended to create a controlled comparison, minimizing potential variables that could influence the results and thereby enhancing the dependability and accuracy of the study's findings (Gülen, 2020).

Following the pilot study, an item analysis was performed to refine the Pre-MAT and Post-MAT tests. Items were evaluated based on their discrimination index and difficulty index, with only those meeting specific criteria being retained. According to Shah et al. (2013), items were kept if they had a discrimination index greater than 0.20 and a difficulty index ranging from 0.30 to 0.70.

The final versions of both the pretest and posttest each consisted of 64 items, including 34 multiple-choice questions (MCQs) and 30 supply-type items (SPI). The average item discrimination index for the pretest (Pre-MAT) was 0.42, while the average difficulty index was 0.40. For the posttest (Post-MAT), the average item discrimination index was 0.44, and the difficulty index was 0.39. These indices are detailed in Table 2. The results indicate that both tests were comparable in terms of difficulty and discrimination, ensuring consistency in the assessment of mathematical achievement.

Table 2. Average item discrimination & item difficulty indexes for pre-post mathematics achievement tests

Mathematics Achievement Test	N	Average Item Discrimination	Average Item Difficulty
Pretest (pre-MAT)	64	0.42	0.40
Posttest (post-MAT)	64	0.44	0.39

Following the analysis of the pilot pre-MAT and post-MAT, a standardized scoring system was implemented, where correct answers were awarded 1 point, and incorrect or blank responses received 0 points. The total scores for each student were then computed, showing a mean score of 34 with a standard deviation of 10.54 for the pretest, and a mean score of 35 with a standard deviation of 11.73 for the posttest.

This comparability in scores between the pre- and post-tests during the pilot phase lends credibility to the study, suggesting that observed differences in scores due to the intervention are likely attributable to the impact of the intervention rather than to inconsistencies in measurement or other extraneous factors (Stratton, 2019; Ruel et al., 2015). This information is further detailed in Table 3. The pre-MAT was administered to both the control and experimental groups before the intervention, while the post-MAT was administered following the intervention.

The Cronbach's alpha reliability coefficients were calculated to ensure the internal consistency of the tests. For the pre-MAT, the reliability coefficient was 0.81, and for the post-MAT, it was 0.85. These values indicate a high level of internal consistency for both tests, suggesting that the items reliably measure the same underlying construct. Table 3 present examples of the items used in the pretest (pre-MAT) and posttest (post-MAT), respectively.

Table 3. Descriptive statistics for piloted pre-post mathematics achievement tests

Mathematics Achievement Tests	N	Mean	*SD*
Pretest (pre-MAT)	243	34	10.54
Posttest (post-MAT)	243	35	11.73

Procedure of Study

The study's participant selection and treatment procedures adhered to ethical and methodological rigor. Approval for the study was granted by the Scientific Research Ethics Committee (SREC) of the relevant institution. Subsequently, fifth graders were divided into two groups: the experimental group, which received instruction using both concrete and virtual manipulatives, and the control group, which followed traditional teaching methods without manipulatives.

The experimental group, consisting of low, average, and high achievers, underwent a 12-week intervention comprising four 45-minute sessions per week, totaling 180 minutes. This group was taught using manipulative-assisted lectures that integrated both physical and digital tools. In contrast, the control group, also stratified by achievement level, received conventional instruction based on traditional teacher exposition methods, delivered by a certified primary school teacher with five years of experience. This approach ensured consistency and minimized potential biases, as emphasized by literature on maintaining study validity (Walter & Max, 2012). Both groups completed the Pre-MAT at the beginning of the intervention and the Post-MAT upon its conclusion. Figure 1 illustrates the treatment design and the overall study procedure.

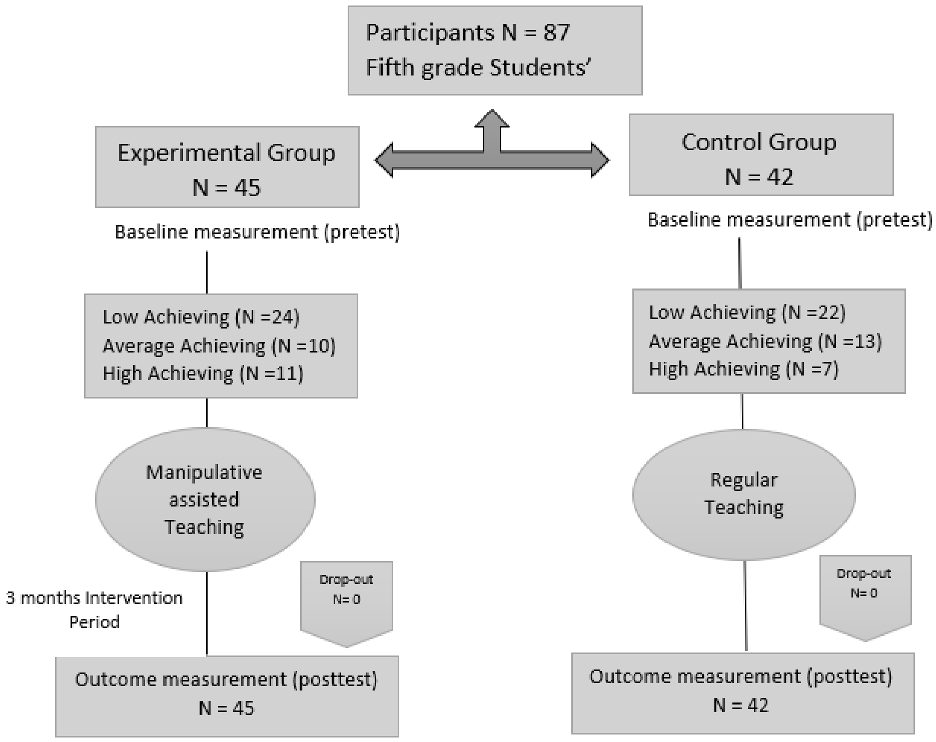

Figure 1. Treatment design

Instructional Settings

The study was conducted in a fifth-grade classroom and a computer lab equipped with 50 computers, each assigned to an individual student to facilitate independent work. The experimental group, comprising low, average, and high-achieving subgroups, engaged with both concrete and virtual manipulatives over a 12-week period in the computer lab. Each session began with the teacher introducing the manipulatives and assigning mathematical tasks, demonstrating their use, and guiding students through the process.

In contrast, the control group, also divided into low, average, and high-achieving subgroups, followed traditional instruction without manipulatives. They attended regular classroom sessions where the teacher presented the day's mathematics topic and assigned drill-and-practice exercises. At the end of each session, the teacher conducted discussions to foster understanding and connect concepts.

The experimental group's sessions were held in the computer lab to ensure convenient access to virtual manipulatives. The students in this group were provided with both concrete and digital tools, which they utilized to complete their tasks independently. This setting allowed the experimental group to integrate and apply both types of manipulatives, whereas the control group continued with conventional teaching methods focused solely on traditional exercises and discussions.

Data Analysis

Mathematics achievement data were collected through pretests and posttests, each consisting of 64 items. The data were analyzed using SPSS 28.0. To assess the distribution of the data, the Kolmogorov-Smirnov test was applied, given that the sample sizes for both the overall groups and their subgroups (low, average, and high achievers) were less than 50 (Kemp, 2021; Pallant, 2020). The results indicated that both the experimental group (low, average, and high achievers) and the control group (low, average, and high achievers) had normally distributed pre- and post-test data ($p > .05$).

Subsequently, parametric tests were employed. To evaluate changes over time for Research Question 1, a paired sample t-test was utilized to compare pre- and post-intervention scores within each test group. For Research Question 2, a one-way analysis of covariance (ANCOVA) was conducted to compare the mathematics achievement of low-achieving students in the control group and the experimental group after the intervention (T2). This test is appropriate for comparing the effects of different interventions when using a two-group pre/post-test design without random assignment, as described by Pallant (2020). Pretest scores were used as covariates to control for initial group differences, aiming to mitigate systematic bias while acknowledging that not all potential differences could be fully eliminated. All relevant assumptions were checked prior to conducting the ANCOVA, including missing data, normality, linearity, outliers, multicollinearity, singularity, homogeneity of variance, homogeneity of regression slopes, and reliability of covariates.

RESULTS AND DISCUSSION

Results for Research Question 1

A paired sample t-test was employed to assess the impact of using both concrete and virtual mathematics manipulatives on fifth-grade students' mathematics achievement across different achievement levels (low, average, and high) from Time T1 (pre-test) to Time T2 (post-test). This statistical method allowed for a comparison of pre- and post-test scores within the experimental groups categorized by achievement levels. The results of these comparisons are summarized in Tables 4, 5, and 6, which detail the performance changes for low, average, and high achievers, respectively.

Table 4. Comparing pre & posttest scores for low achiever experimental group students

Variables	N	Mean	SD	df	t-value	Sig.	Cohen's d
Pretest Scores	24	22.83	1.49	23	-21.723	<0.001	0.95
Posttest Scores	24	40.37	4.18				

Table 4 illustrates that there was a significant improvement in mathematics achievement for low achievers in the experimental group. The mean score increased from 22.83 (SD = 1.49) on the pre-test to 40.37 (SD = 4.18) on the post-test. The paired sample t-test yielded a t-value of -21.723 with a p-value less than 0.001, indicating a highly significant change. The mean increase of 17.54 points falls within a 95% confidence interval of -19.21 to -15.87. The eta-squared value of 0.95 suggests a large effect size, demonstrating that the intervention had a substantial impact on the mathematics achievement of low achievers, leading to a significant improvement in their post-test scores.

Table 5. Comparing pre & posttest scores for average achiever experimental group students

Variables	N	Mean	SD	df	t-value	Sig.	Cohen's d
Pretest Scores	10	31.50	3.53	9	-7.99	<0.001	0.87
Posttest Scores	10	42.10	4.88				

Table 5 demonstrates a significant improvement in mathematics achievement for average achievers in the experimental group. The mean score increased from 31.50 (SD = 3.53) on the pre-test to 42.10 (SD = 4.88) on the post-test. The paired sample t-test revealed a t-value of -7.99 with a p-value of 0.00, indicating a significant change. The mean increase of 10.60 points falls within a 95% confidence interval of -13.60 to -7.59. The eta-squared value of 0.87 suggests a large effect size, indicating that the intervention had a substantial positive impact on the mathematics achievement of average achievers, resulting in a significant enhancement in their post-test scores.

Table 6. Comparing pre & posttest scores for high achiever experimental group students

Variables	N	Mean	SD	df	t-value	Sig.	Cohen's d
Pretest Scores	11	50.27	1.48	10	-8.11	<0.001	0.86
Posttest Scores	11	57.63	3.69				

Table 6 illustrates a significant improvement in mathematics achievement for high achievers in the experimental group. The mean score increased from 50.27 (SD = 1.48) on the pre-test to 57.63 (SD = 3.69) on the post-test. The paired sample t-test yielded a t-value of -8.11 with a p-value of 0.00, indicating a statistically significant difference. The mean increase of 7.36 falls within a 95% confidence interval of -9.38 to -5.34. The eta-squared statistic of 0.86 indicates a large effect size, signifying that the intervention had a substantial positive impact on the mathematics achievement of high achievers. This demonstrates that the post-test scores for high achievers were significantly higher compared to their pre-test scores, highlighting the effectiveness of the intervention in enhancing their mathematics performance.

Descriptive Statistics

The findings presented in Table 7 highlight the differences in mathematical achievement between low-achieving students in the control and experimental groups. This table shows substantial changes in mean scores and standard deviations for both pre-tests and post-tests, underscoring the effect of the different teaching methodologies applied.

Furthermore, Table 8 provides a detailed statistical analysis using one-way ANCOVA to further examine the significance of these changes. This analysis helps to clarify the extent to which the observed differences in achievement scores can be attributed to the intervention rather than pre-existing differences between the groups. By controlling for pre-test scores as covariates, the ANCOVA ensures a more accurate assessment of the intervention's impact, providing a comprehensive understanding of how the manipulatives-based instruction compared to traditional methods in enhancing the mathematical performance of low-achieving students.

Table 7. Mean Performance Scores of Low Achievers Classified by Instructional Strategy/ Groups

Instructional Groups	Pretest		Posttest		Mean Gain	
	N	M	SD	M	SD	M_g
Control	22	22.18	2.038	23.68	2.868	1.50
Experimental	24	22.83	1.493	40.37	4.189	26.12

The data in Table 7 elucidates the impact of different teaching methodologies on the mathematical achievement of low-achieving students. In the control group, which employed traditional teaching methods, there was a modest improvement in mean scores over the intervention period. Specifically, the mean score increased from M(pre) = 22.18 to M(post) = 23.68, reflecting a minimal enhancement in mathematical achievement. The standard deviation values for pre-test and post-test were SD = 2.038 and SD = 2.868, respectively, indicating limited variability in scores and suggesting that improvements were not widespread among students.

In contrast, the low-achieving students in the experimental group, who received instruction using mathematical manipulatives, exhibited a substantial improvement in their mean scores. The mean scores increased from M(pre) = 22.83 to M(post) = 40.37, demonstrating a significant positive effect of the manipulative-based intervention on their mathematical achievement. The standard deviation values for pre-test and post-test were SD = 1.493 and SD = 4.189, respectively. These values highlight a greater dispersion in post-test scores, reflecting a broader range of improvements among students and indicating that the manipulatives-based instruction was effective in enhancing achievement across the low-achieving group.

Results for Research Question 2

A one-way analysis of covariance (ANCOVA) was employed to evaluate the effect of mathematics manipulatives (both concrete and virtual) on the post-test scores of low-achieving fifth-grade students, relative to a control group that did not utilize manipulatives. The pre-test scores served as a covariate in this analysis to account for baseline differences between the groups. The findings from this analysis are detailed in Table 8.

Table 8 presents the results of a one-way Analysis of Covariance (ANCOVA), specifically focusing on a between-group analysis designed to evaluate the comparative efficacy of two distinct interventions in enhancing mathematical achievement among fifth-grade students. The independent variable in this analysis was the type of intervention, distinguishing between traditional teaching methods and instruction incorporating mathematics manipulatives. The dependent variable was the scores achieved on the post-intervention mathematics achievement test (post-MAT), administered following the completion of the interventions.

Table 8. One-Way ANCOVA of posttest scores for low achieving control and experimental groups, with pretest scores as covariate

Source	df	Type III SS	MS	F	p	η^2
Intercept	1	211.530	211.530	16.899	<0.001	.282
Covariate (Pretest Achievement)	1	38.150	38.150	3.048	0.088	0.06
Groups (Control/Experimental)	1	808.181	808.181	64.565	<0.001	0.600
Error	43	538.248	12.517			
Total	46	52038.000				

R Square = 0.857 (Adjusted R Square = 0.851)

To enhance the accuracy of the analysis, pre-intervention scores on the mathematics achievement test (pre-MAT/pretest) for both control and experimental groups of low-achieving students were included as a covariate. Preliminary checks were conducted to ensure that the assumptions outlined in the data analysis section were met.

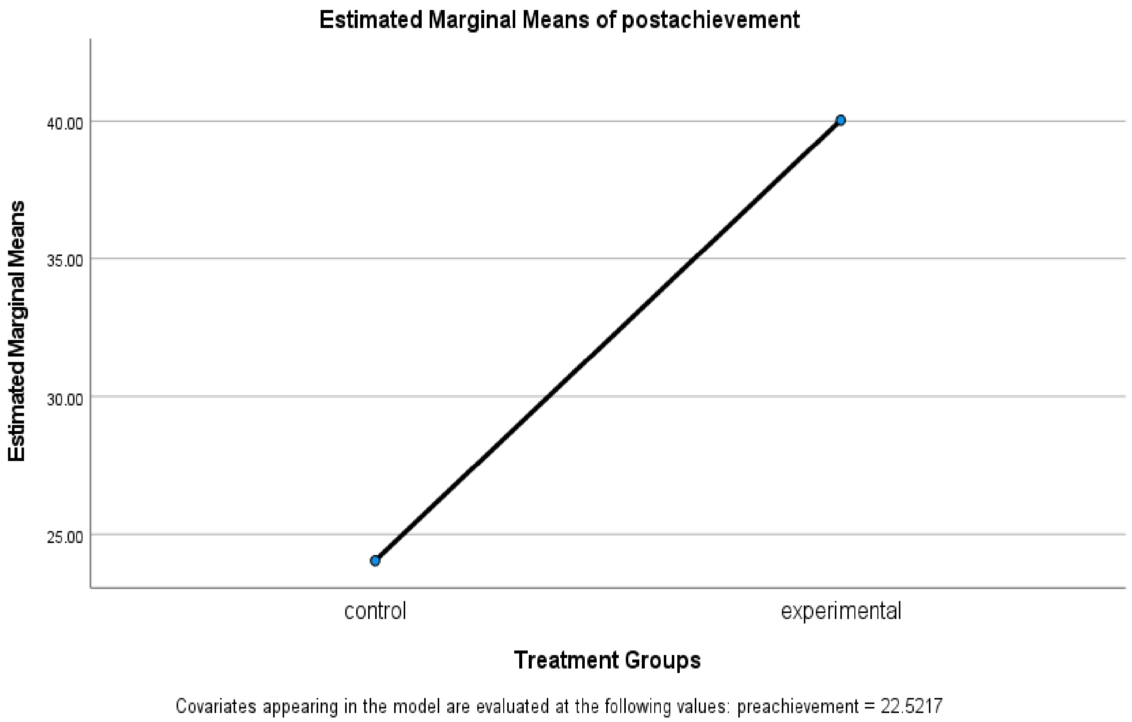

Figure 2. Comparing posttest scores for control & experimental groups (low achievers)

After adjusting for pre-intervention (pretest) scores, a statistically significant difference was observed between the control and experimental groups regarding post-intervention (posttest) scores on the mathematics achievement test. This was evidenced by an F-value of 64.565 with 1 and 43 degrees of freedom ($F(1, 43) = 64.565$, $p < 0.001$), and a large effect size indicated by a partial eta squared of 0.600. This result underscores the substantial impact of the interventions on posttest scores.

It is important to note that the relationship between pretest and posttest scores on the mathematics achievement test was minimal, as indicated by a partial eta squared value of 0.06. This suggests that pretest scores had a limited effect on posttest performance, reinforcing the effectiveness of the interventions in achieving significant improvements in mathematical achievement. Specifically, students in the experimental group (M = 40.37, SD = 4.189) demonstrated a notable improvement in their posttest scores compared to students in the control group (M = 23.68, SD = 2.86) due to their participation in the math-manipulative-assisted instruction. For additional details, refer to Table 7. Figure 2 provides a visual representation supporting this interpretation.

Discussion

The primary objective of this study was to investigate the impact of using both concrete and virtual manipulatives on the mathematics achievement of fifth-grade students across varying achievement levels (low, average, and high). To achieve this objective, two research questions were formulated. These questions were explored using a quasi-experimental research design, and the data were analyzed through independent sample t-tests and paired sample t-tests. The results of this analysis are detailed in the preceding section, supported by graphical representations.

In the following paragraphs, we will discuss how the findings relate to existing literature and how they contribute to the broader understanding of the topic. Additionally, a subsequent section will address the study's limitations, present comprehensive conclusions, and offer recommendations for future research.

The findings of this study indicate that the intervention incorporating virtual manipulatives preceding concrete manipulatives effectively improved the mathematics achievement of students across all subgroups (low, average, and high achievers) in the experimental group. This was evident from the significant improvement in post-intervention test scores compared to pre-intervention scores (Research Question 1). The data demonstrate that students at different achievement levels who participated in the intervention showed greater gains in their post-test scores compared to their pretest scores.

Furthermore, a one-way ANCOVA, conducted after verifying the assumptions of the test, revealed that low-achieving students in the experimental group, who received manipulative-assisted instruction, performed significantly better than their counterparts in the control group, who were taught using a traditional deductive teaching method (Research Question 2). This suggests that the experimental group's post-test scores were substantially enhanced as a result of the intervention.

These results imply that the integration of concrete and virtual manipulatives has a significant impact on improving students' mathematical achievement in areas such as number systems, geometry, and information handling. One possible explanation for these improvements is that the combination of concrete and virtual manipulatives provides students with both visual and practical insights into abstract mathematical concepts, thereby facilitating the development of the conceptual understanding necessary to grasp these underlying abstract principles.

The findings of this study align with previous research in mathematics education, including the works of Toth et al. (2009), Zacharias and Olympious (2011), and Zacharias & de Jong (2014). These studies demonstrated that the use of a combination of manipulatives—regardless of their sequence, whether virtual preceding concrete or vice versa—can be more beneficial for enhancing the achievement of primary grade students (fourth and fifth grades) and undergraduate students in understanding abstract science concepts compared to the use of no manipulative-assisted instruction.

A distinguishing feature of this study is its demonstration that the benefits of combining concrete and virtual manipulatives extend to mathematics education, similar to their effects in science-related subjects. This may be attributed to the incorporation of mathematical simulations, which aimed to facilitate students' visualization and experiential understanding of underlying mathematical processes. Such simulations are instrumental in helping students construct a robust conceptual understanding of number systems, geometry, and information handling.

It has been argued that practical models can be challenging for young students due to the abstract nature of mathematical concepts (Bouck et al., 2023; Nurjannah & Kusnandi, 2021; Sugiarti & Retnawati, 2019; Sulistiowati et al., 2019). However, other researchers have posited that practical models can significantly aid primary-age students in understanding mathematical concepts and improving their academic achievement (Shuxratovna, 2024; Bone et al., 2023; Ukdem & Çetin, 2022).

In this context, the current study's findings corroborate the conclusions of Wilkie and Hopkins (2024), Shen (2023), Back (2019), and Tjandra (2023). This study argues that fifth-grade students can begin to comprehend practical representations of whole numbers, fractions, decimals, percentages, unitary methods, geometry, and perimeter & area. The results suggest that practical models serve as a powerful tool for developing an understanding of abstract or complex mathematical concepts and enhancing mathematical achievement, particularly for fifth-grade students.

CONCLUSION

The findings of this study reveal a statistically significant enhancement in the post-test mean scores of the experimental group of students, categorized by achievement levels (low, average, and high), compared to their pre-test scores. This outcome suggests that the integration of concrete and virtual manipulatives in mathematics instruction effectively improves the mathematical achievement of fifth-grade students across various achievement levels. While the low-achieving students in the control group exhibited some improvement in mathematics achievement relative to their pre-test scores, the effect size was moderate. Notably, the low-achieving students in the experimental group demonstrated superior performance compared to their counterparts in the low-achieving control group. This indicates that the application of both concrete and virtual manipulatives is more efficacious than traditional teaching methods in enhancing mathematical achievement.

Several limitations of the study must be considered when interpreting these results. Firstly, potential teacher bias could have influenced the effectiveness of the manipulatives, as the success of their use may be contingent upon the teacher’s proficiency. Variability in teachers’ skills with manipulatives may introduce confounding factors. Additionally, external factors beyond the scope of the study, such as students' home environments or personal issues, could also impact their academic performance and affect the generalizability of the findings.

Future research should address these limitations and explore several avenues to build upon the current study. First, longitudinal studies are recommended to assess the enduring effects of concrete and virtual manipulatives on mathematical achievement over extended periods. This would provide insight into whether the observed benefits are sustained over time. Second, investigations into the optimal combination of concrete and virtual manipulatives are warranted to determine the most effective ratio for instructional purposes. Comparative studies could also evaluate the impact of using each type of manipulative independently versus in combination. Third, research could examine the differential impact of manipulatives on specific student subgroups, such as those with learning disabilities or diverse cultural backgrounds, to ascertain the universality of the benefits. Lastly, studies exploring the feasibility and efficacy of implementing manipulatives on a larger scale, such as across entire schools or districts, would help determine whether the observed benefits are replicable and whether the costs associated with broader implementation are justified by the outcomes.

Declarations

Author Contribution	:	SA: Conceptualization, Writing - Original Draft, Methodology, Visualization, Editing, and Formal analysis H-SS: Validation and Supervision
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