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Constructing slope conceptualizations: Physical, geometrical, and algebraic

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Abstract

The concept of slope constitutes a fundamental component of the discourse surrounding linear equations. A subset of students frequently interprets slope merely as an algebraic ratio. This particular context fosters a superficial understanding of slope, as these students typically resort to mechanical memorization of the slope formula. The intent of this research endeavor is to enhance a holistic understanding of the slope concept: physically, geometrically, algebraically; through the deployment of realistic teaching activities. Two students are participants in this research endeavor. The research group initiated a series of questions aimed at assessing their comprehension by delivering a total of six activities, which were systematically designed using the emergent modeling framework central to the educational design principles of Realistic Mathematics Education (RME). These activities are structured to facilitate the students' understanding of the fundamental concept of slope, transitioning from physical properties, through geometric ratios, to algebraic ratios. Subsequent to the investigations and the interviews conducted, the researchers deduce that the utilization of realistic activities significantly enhances students' comprehension of the foundational concept of slope: physical, geometric, algebraic.

Keywords: Algebraic Ratio, Geometric Ratio, Physical Property, Slope

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The concept of slope serves as a fundamental mathematical construct in educational curricula worldwide, acting as a prerequisite for the understanding of more advanced concepts in mathematics, chemistry, biology, and physics (Nagle & Moore-Russo, 2014). It is considered a foundational element in the teaching and learning of linear equations, with high school students frequently tasked with calculating, estimating, and interpreting the gradient of linear functions (Planinic et al., 2012). Slope is typically characterized by terms such as steepness, elevation, descent, or inclination (Deniz & Kabael, 2016). Existing research identifies three primary representations of slope: its application in physical contexts, such as in the design of mountain roads, ski slopes, and wheelchair ramps; its use in functional contexts, such as the relationship between distance and time (Stump, 2001); and its graphical representation, commonly encountered in coordinate geometry (Lingefjärd & Farahani, 2018).

The formal development of slope conceptualization typically begins in the middle grades (Hoffman, 2015). An interview with an Indonesian mathematics teacher revealed that both students and educators in Indonesia face significant challenges in understanding the concept of slope. This difficulty is partly attributed to the Indonesian mathematics textbooks, which primarily focus on the algebraic ratio and require students to calculate the slope by dividing the difference in y-values by the difference in x-values (Depdikbud, 2017). These observations align with international research highlighting similar difficulties students encounter with slope concepts. For example, both students and teachers often struggle to apply the concept of slope across various contexts (Stump, 1999), with a tendency to prioritize procedural calculations over deeper understanding (Lingefjärd & Farahani, 2018). Additionally, students frequently face challenges in translating graphical representations into equations (Planinic et al., 2012) and tend to memorize the slope formula without understanding the underlying relationship between the rate of change, the dependent variable, and the independent variable (Deniz & Kabael, 2017). Further complicating this issue, students also encounter difficulties in representing slope both as a geometric ratio and as an algebraic ratio (Ornekci & Cetin, 2021). In a similar vein, a teacher interviewed for this study noted that students' understanding of slope is often limited to its algebraic ratio representation. Consequently, when asked to consider other forms of slope representation, students frequently experience confusion.

Several strategies have been proposed to assist teachers in helping students construct and understand the concept of slope. Some researchers advocate for the use of multiple representations to support the development of diverse ways in which students can grasp the idea of slope (Lingefjärd & Farahani, 2018; Nagle & Moore-Russo, 2014). Others suggest that incorporating real-world situations is crucial for providing a meaningful context for teaching the concept of slope (Carreira & Baioa, 2016; Crawford & Scott, 2000; Nagle & Moore-Russo, 2014; Stanton & Moore-Russo, 2012). This study aims to bridge the gap between real-world situations and the various representations of slope.

The focus of this paper is on the teaching and learning of the concept of slope in Indonesian middle schools. To guide this process, we adopt a Realistic Mathematics Education (RME) approach, which facilitates the transition from informal to formal mathematical thinking through the application of real-world contexts (Deniz & Kabael, 2016). Within the framework of RME, we employ emergent modeling as our instructional design. It is widely recognized that students require more than a procedural understanding of slope as an algebraic ratio; they must develop a deeper conceptual understanding that integrates its physical properties and geometric ratio representations. In this study, we aim to support students in developing a comprehensive understanding of slope, encompassing physical properties, geometric ratios, and algebraic ratios. We hypothesize that engaging students in realistic mathematical activities will foster a more profound grasp of the fundamental concept of slope.

Slope

The concept of slope, as related to linear equations, can be described in various ways, including steepness, slant, or the incline of a route, whether uphill or downhill. It is applied in numerous fields such as geography, architecture, and programming. Stump (1999) categorized slope into several types, including: a geometric ratio (rise over run), an algebraic ratio $\frac{(y_2-y_1)}{(x_2-x_1)}$, a physical property (steepness or incline), a functional property (rate of change between two variables), a parametric concept (the slope coefficient m in $y = mx + b$), a trigonometric concept (the tangent of the angle of inclination), and a calculus concept (the derivative). Moore-Russo et al. (2011) further expanded this classification by adding three additional categories: the determining property (where slope determines parallelism and perpendicularity), slope as a behavior indicator (which reveals whether the line is increasing, decreasing, or horizontal), and the linear constant property (where the slope remains unchanged under translation of the line).

However, the Indonesian junior high school mathematics curriculum (Depdikbud, 2017) lacks these comprehensive categorizations of slope. The curriculum only outlines three objectives: graphing a linear equation, determining the slope or gradient of a line, and solving linear equations. In practice, students are primarily asked to calculate the algebraic ratio for determining the slope of linear equations. We argue that for a more profound understanding, students must first grasp the physical property and geometric ratio of slope before exploring the algebraic ratio, as these conceptualizations serve as foundational prerequisites.

Difficulties in Understanding Slope

A substantial body of research has explored the complexities involved in understanding the concept of slope. As highlighted in the introductory section, several challenges in this area have been identified. In addition to these difficulties, learners often struggle to differentiate between the slope and the elevation of a graph, and they face challenges in interpreting variations in both slope and elevation (McDermott et al., 1987; Planinic et al., 2012). Ornekci and Cetin (2021) outlined several common misconceptions related to slope, including: the misinterpretation of horizontal distance as slope without considering the corresponding vertical change; the erroneous belief that slope increases as the angle decreases; confusion between the direction of the line and the slope; difficulties in accurately identifying points on the coordinate plane; and a lack of understanding of the set of rational numbers. Additionally, students frequently struggle with the Cartesian coordinate system, often failing to apply the correct scale along both the x-axis and y-axis. A subset of students mistakenly represents the x-axis as a vertical coordinate and the y-axis as a horizontal coordinate. Furthermore, many students exhibit confusion regarding positive and negative values on the number line, which can hinder their grasp of the fundamental concept of slope. These misunderstandings may lead students to view slope merely as a formula, devoid of real-world relevance. However, the concept of slope is crucial in fields such as construction and architecture, where its application is essential for ensuring both functionality and safety.

RME Approach

Students' everyday experiences can serve as an effective bridge to mathematical concepts (Bonotto, 2013; Carreira & Baioa, 2016). RME approach is specifically designed to facilitate this connection. As one of several pedagogical methodologies for mathematics instruction, RME is grounded in five core principles: phenomenological exploration, the use of vertical instruments, student-driven construction and production of knowledge, interactive engagement, and the integration of various mathematical domains (Treffers, 1987; Wijaya, 2008). In addition, Gravemeijer and Bakker (2006) and Wijaya (2008) further elaborate on RME by proposing three heuristic frameworks: guided reinvention, didactical phenomenology, and emergent modeling.

Guided reinvention underscores the importance of learners "reconstructing" mathematical concepts and methods, thus supporting the transition from informal to formal mathematical reasoning. This process is driven by students' independent problem-solving efforts, guided by carefully designed instructional strategies (Gravemeijer, 1994). Didactical phenomenology highlights the need to select relevant contexts or phenomena that inherently lead students to discover significant mathematical structures (Freudenthal, 1991). Emergent modeling describes the progression through which students' initial informal representations of a specific context (model-of) evolve into more generalized mathematical tools or frameworks (model-for), thereby enhancing the connection between concrete experiences and abstract cognitive processes (Gravemeijer, 1999).

In this study, we aim to leverage practical, real-life experiences as a contextual foundation to design a series of activities intended to deepen students' understanding of slope. These activities are framed within a Hypothetical Learning Trajectory (HLT) model (Simon, 1995).

METHODS

This study utilized a design-based research methodology comprising three distinct phases: preliminary design, instructional experiments, and retrospective analysis (Simon et al., 2018). In the preliminary design phase, a theoretical learning trajectory was developed, encompassing three key components: learning objectives, learning activities, and students' conjectures regarding these activities (Simon, 1995). The trajectory was constructed in alignment with the emergent modeling framework derived from RME approach.

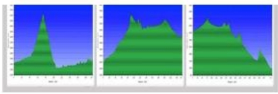
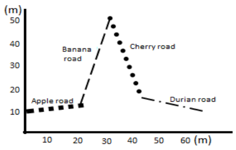
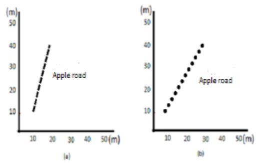
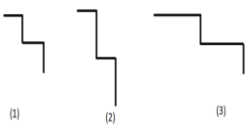
During the instructional experiment phase, two eighth-grade students were engaged in solving a problem related to their conceptual understanding of slope. The students were tasked with exploring various aspects of slope, with particular emphasis on its geometric and contextual representations. Furthermore, in the retrospective analysis phase, data from student interviews were examined to assess their investigative processes and responses, shedding light on their evolving understanding of slope.

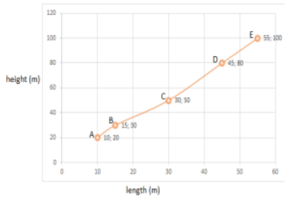
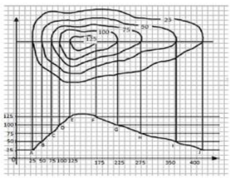
The emergent modeling instructional design framework, which guided this investigation, is structured into four levels: situational, referential, general, and formal (Gravemeijer, 1994; 1999). The situational level represents the initial stage of emergent modeling, wherein students apply specific knowledge and strategies relevant to the given context. In this study, students analyzed pictorial representations of bike race routes during Activity 1, focusing on the steepness of the routes. The referential level involves the construction of informal models or representations that are closely tied to the context, reflecting students' experiences with slope (Gravemeijer, 1994). In Activity 2, students were presented with images comparing different roadways, allowing them to observe and compare attributes related to steepness. Activity 3 was designed to facilitate students' understanding of the significance of the x-axis (length) and y-axis (height) within the context of slope.

The general level (transitioning from model-of to model-for) represents the stage at which informal representations begin to evolve into more abstract conceptual frameworks for understanding slope (Gravemeijer, 1999). Through successive activities, students developed an understanding of the significance of the x-axis and y-axis. In Activity 4, they used the vertical and horizontal components of a staircase as a model-of to determine the safest staircase. In Activity 5, students transitioned to employing the x- and y-axes as a model-for to perform formal slope calculations. The formal level marks the culmination of the emergent modeling process, in which students apply formal mathematical representations to abstract and consolidate their understanding of slope (Treffers, 1987). In this final phase, students were tasked with determining the safest route based on formal slope calculations. All activities are represented within the HLT for teaching the concept of slope, as detailed in Table 1.

The study was conducted at a junior high school in Java, Indonesia, during the "new normal" era, when in-person learning sessions were conducted under time constraints. Participants were selected based on teacher recommendations, and approval for the research was obtained from the school's headmaster. Both the teacher and students provided informed consent to participate in the study. Data were gathered through unstructured interviews, and the qualitative responses from students were analyzed using retrospective analysis to gain insights into their conceptual understanding of slope.

Table 1. Overview of the HLT for learning the concept of slope

Activity	Learning Goal	Description of Activity	Conjecture
Routes for bike-race 	Students can understand the physical property and relate it to 'steepness,' 'up-hill,' 'down-hill' of the road	To choose and to justify the best route for the bike race	Students might limit their justification to visual elements, Students might limit their justification to the number of downhill roads
Comparing non-identical roads 	Students can analyze the physical property related to height and length	To justify the steepness based on the height of the graph by seeing the y-axis	Students might relate the steepness of the road to the y-axis, Students might analyze the steepness of the road related to the image
Comparing identical roads 	Students can understand slope as a geometric ratio	To justify the steepness not only based on the height (y-axis) but also the length (x-axis)	Students might understand that the slope of a road depends on the y-axis and the x-axis, Students might not realize that the B road is higher than the A road
Comparing staircases 	Students can apply the geometric ratio as 'rise over run.'	To choose the safest staircase and make calculations regarding to its steepness	Students might choose the staircase which has a more extended tread than its mirror; they will apply slope value as a geometric ratio (vertical side over the horizontal side), Students might choose the staircase which has thread = mirror
Comparing some points of the roads	Students can apply the algebraic ratio	To calculate the slope at two different points of the graph	Students might consider the ratio of the vertical distance (y-

Activity	Learning Goal	Description of Activity	Conjecture
			axis) and the horizontal distance (x-axis), Students might merely apply the formula: (y-axis) over the length (x-axis) of the road to find the slope
Comparing top-view and side-view of the hill 	Students can apply the algebraic ratio	To analyze the safest route by using algebraic justification	Students might consider the ratio of the vertical distance (y-axis) and the horizontal distance (x-axis), Students might merely apply the formula: (y-axis) over the length (x-axis) of the road to find the slope

RESULTS AND DISCUSSION

The analysis of the research findings presented in Table 1 demonstrates the gradual development of students' understanding of the concept of slope through a series of meticulously structured learning activities. The results indicate that while students exhibit progress in grasping the concept of slope across the various activities, further focus on the relationship between vertical and horizontal distances within both geometric and algebraic contexts is necessary for a more thorough and comprehensive understanding of the concept. A detailed explanation of each activity in the study will be discussed further as follows.

Activity 1: Routes for Bike-Race

The activity 'Routes for Bike-Race' evaluates students' understanding of the physical properties of the slope concept. In this activity, students are tasked with determining which of the three mountain peaks provides the optimal and safest route for a bike race, as illustrated in Figure 1. The participating students are assumed to have a foundational understanding of prerequisite concepts, including ratio and basic algebra. The gradient is expressed as an algebraic relationship between two variables, requiring students to simplify equations, solve for unknowns, and understand linear relationships (Kieran, 1992). The gradient is defined as the ratio of the change in the dependent variable (Y-axis) to the change in the independent variable (X-axis), often referred to as "rise over run." A solid understanding of ratios and comparisons is essential for students to interpret the numerical meaning of the gradient (Van de Walle et al., 2019). Given that distance is represented on the X-axis and elevation on the Y-axis, students are expected to estimate and compare the slopes of the different mountains. However, it is anticipated that students may primarily rely on a visual interpretation of the slope, focusing their arguments on whether the roads incline or decline. The physical property concept serves as an introductory framework for understanding slope in broader mathematical contexts. The use of various race-track representations provides a practical and engaging context to foster a deeper understanding of the slope concept (Deniz & Kabael, 2016). In our interview, Vascha selected the third image depicted in Figure 1.

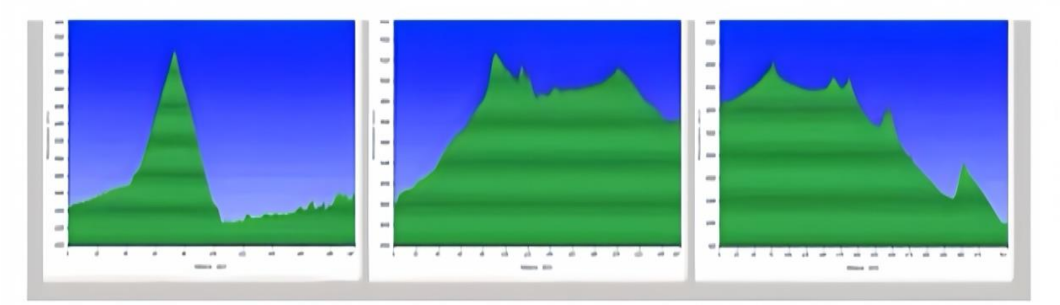


Figure 1. Routes for a bike race (source: https://www.worldtravellerss.be/asiatrails/mainpaginas/maintajikistan)

The following is an excerpt from the interview between the teacher (A) and Vascha, in which Vascha explains the reasoning behind the selection of the image:

A : Which trail do you think is the best for bike racing? Could you explain why that trail is the best?
Vascha : The third picture, Miss, because it has a steep downhill section.
A : What do you mean by "down-steep," Vascha?
Vascha : The route is going downhill.
A : Does the shape of the road influence your decision?
Vascha : When the road is straight, the ride is steady. But when the road is zig-zagged, it becomes unsteady.

Furthermore, Figure 2 provides an illustration that explains Vascha's response concerning the relationship between road conditions and bicycle speed.

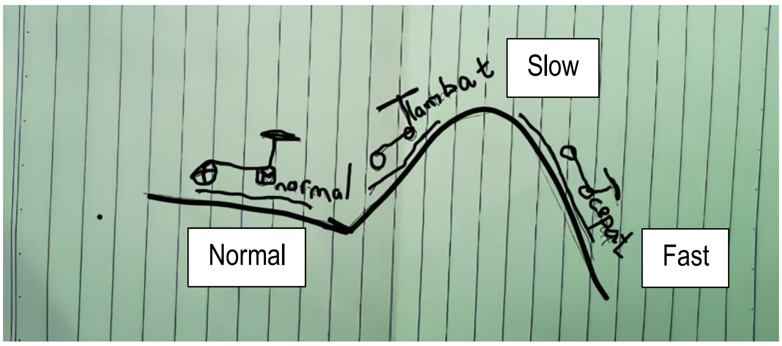


Figure 2. Vascha's shows in a drawing the relation between the road and the bike's speed

Indah approached the question differently. Initially, she considered whether the bike race would proceed from left to right or from right to left.

A : Which trail do you think is the best for bike racing? Please explain why that trail would be the best.
Indah : Which direction will the race start, Miss?
A : What if it starts from the left side?
Indah : The second picture.

A : Why do you think the second picture would be the best trail?

Indah : It's because of the starting position, Miss. If the race begins from the left, the starting point of the second trail is lower than that of the third. But if it starts from the right side, the third picture is lower.

A : Could you elaborate on the more challenging and easier sections of the trail in the second picture?

Indah : The second picture represents a safer road because it only has one uphill section (which is the most challenging part), and the middle section is nearly flat, making it easier to navigate. The downhill part is also relatively easy, though you would need to use the brakes while riding.

A : Why didn't you choose the first picture? The trail also starts from left to right, or from right to left, with a downhill section.

Indah : Yes, but the middle part is dangerous. The uphill section is longer, and the downhill section is longer as well, which makes it seem like a narrow mountain pass.

A : What do you mean by "the uphill section is longer"?

Indah : The track, Miss, is very steep.

A : What do you think would happen if the track's slope were less steep?

Indah : It would be safer to navigate, Miss.

Activity 2: Comparing Non-Identical Roads

The Comparing Non-identical Roads activity assesses students' understanding of the physical property of slope in relation to the y-axis. In this activity, students were presented with various road profiles, each accompanied by information regarding their length and elevation. The objective was for students to analyze the steepness of each road in relation to the y-axis. However, their analysis may have been influenced by visual representations of the roads or by focusing primarily on the height differences. In a previous interview (Activity 1), Indah used terms such as "narrow mountain" and "so steep" to describe certain roads as shown in Figure 3.

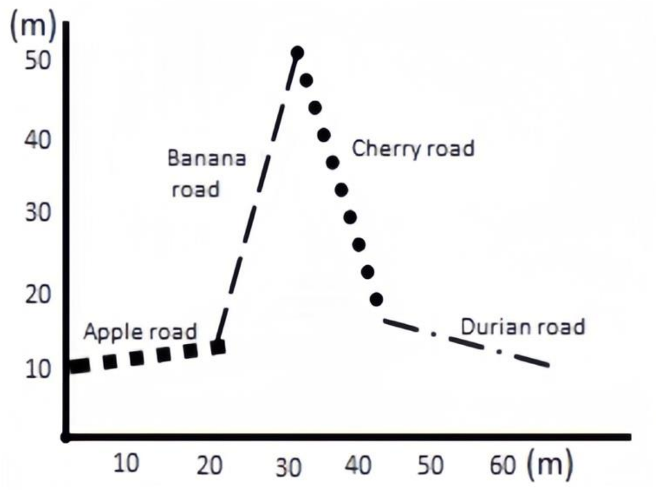


Figure 3. Non-identical road

Furthermore, an interview was conducted with the aim of exploring Indah's understanding of the components of steepness, specifically how it relates to changes in elevation along the y-axis.

A : I recreated the bike racetrack that you didn't choose yesterday, with a road in the middle that you referred to as dangerous due to its length. What do you mean by that?
Indah : It's dangerous because of the length of the Banana and Cherry roads. The distance means long uphill, Miss.
A : The uphill is long. Could you explain further?
Indah : Because it goes uphill and it's long.
A : Is the Banana Road longer than the Apple Road?
Indah : The Apple Road is 20, while the Banana road's distance is from 20 to 30, so it's only 10. Therefore, the Apple Road is longer.
A : Does this mean the Banana Road is short?
Indah : No, the Banana Road has a shorter distance, but the road is steeper or higher than the Apple Road.
A : The distance is short but high. Could you elaborate on that?
Indah : Yes, the distance is from 20 to 30, which is 10, but the height goes from 10 to 50, so the difference in height is 40.

In this interview, we recreated the road that Indah avoided in Activity 1, providing additional information in the hope that she would consider it in her analysis. Indah replaced the term "steep" with the description of the uphill section as "long." By asking her to compare the Banana Road and the Apple Road, we aimed to gain more insight into her understanding of the concept of steepness. Indah focused on the length and height differences, noting that while the Apple Road appeared to be longer, the height difference for the Banana Road was greater. Indah seemed to conceptualize "distance" as corresponding to the x-axis and "height" as the y-axis.

A : If I were to ask you to bike on all of these roads, would you be willing to do so?
Indah : I would be willing to ride on the Apple Road and the Durian Road because they are long and not very steep, making them safer to ride on. But I would not want to bike on the Banana and Cherry roads, because they are very steep. The length is short, but the height is very high, which makes them hazardous, as the roads form almost a vertical line.
A : A vertical line? Could you explain what you mean by that?
Indah : Yes, the vertical line represents the height. The path goes straight up, making it dangerous.
A : So, what would you consider the safest type of road?
Indah : A flat road, one that is horizontal, like a flat line. It represents the distance traveled, or the length of the road.

Activity 3: Comparing Identical Road

The Comparing Identical Roads activity directs students' attention to the physical property of slope in relation to the x-axis. In this activity, two road profiles with identical height differences but different lengths are presented. Students are tasked with analyzing the steepness of the roads and relating it to the x-axis. In a previous interview (Activity 2), Indah referred to the vertical line as representing height and the horizontal line as representing length as shown in Figure 4.

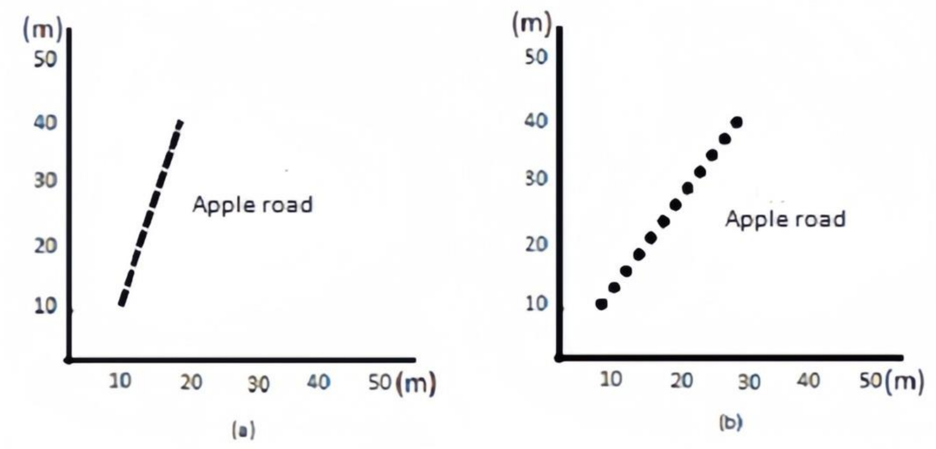


Figure 4. Identical road

Furthermore, an interview was conducted to explore Indah's understanding of the components of steepness, particularly in relation to distance, or the x-axis.

A : Look at these pictures. Can you compare the roads?
Indah : The road in the first picture is higher, Miss.
A : Why do you think that is?
Indah : Because it is uphill, and the incline is very steep.
A : Interesting. So, you consider the first road to be higher because it has a steep incline, is that correct? Could you explain that further?
Indah : The first road is short, only from 10 to 20.
A : What does that imply?
Indah : The shorter the distance, the steeper the road.
A : Can you compare this with the second picture?
Indah : The second road has a longer distance, from 10 to 40, so it is not as steep as the first road.
A : Let me ask you further, how much does the Apple Road climb in the first and second pictures?
Indah : The road in the first picture climbs from 10 to 40, which is the same as in the second picture.
A : You mentioned that the Apple Road in the first picture is higher, but you also said both roads climb the same distance. Can you explain what is happening there?
Indah : Yes, Miss. The height is the same, but the first road is shorter in length compared to the second road. So, the Apple Road in the first picture looks higher.
A : Can you now explain the difference between the two images?
Indah : Although the height of the road is the same, the length of the distance is different. The Apple Road in the second picture is safer because the distance is longer than in the first picture.
A : Can you explain how horizontal distances relate to the height?
Indah : Even though the road is steep, if the distance is long, the road will be safer. But if the road is high and the distance is short, then the road will be very steep.

Activity 4: Comparing Staircase

The Comparing Staircase activity is designed to enhance students' understanding of the geometric ratio concept of slope. Building upon previous activities (Activity 3), the students have already grasped the concept of slope, particularly in terms of the physical properties related to the x-axis and y-axis. Indirectly, Indah recognized that both the x-axis (distance or length) and the y-axis (height) affect the steepness of a road. In this activity, we presented three different types of staircases: Staircase 1 (where the tread and riser are of equal length), Staircase 2 (where the riser is twice the size of the tread), and Staircase 3 (where the tread is twice the size of the riser) as shown in Figure 5. Students were asked to choose the safest staircase based on these conditions. The riser-tread ratio plays a significant role in determining stair safety and comfort. We expected students to calculate the slope of their chosen staircase using the "rise over run" method.

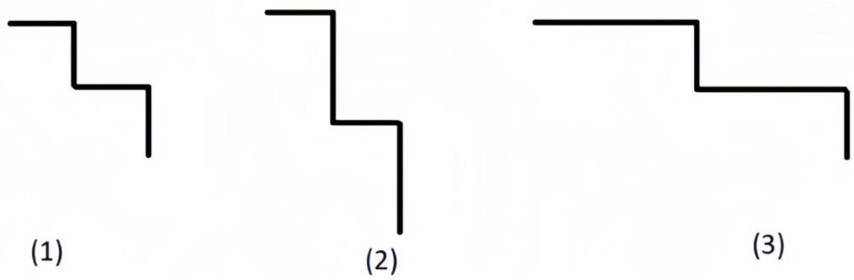


Figure 5. Staircase

Furthermore, an interview was conducted with the aim of exploring Indah's understanding of the geometric ratio concept of slope using several staircases.

A : There are three staircases: Staircase 1 (with equal tread and riser lengths), Staircase 2 (where the riser is twice the length of the tread), and Staircase 3 (where the tread is twice the length of the riser). Which of these staircases do you think is the safest?

Indah : Staircase 3, because we only need to take a few steps, so it requires less energy to walk.

A : Could you compare these three staircases based on your reasoning for safety?

Indah : Staircase 1 is fairly safe, Staircase 2 is probably not safe, and Staircase 3 is very safe.

A : Why do you consider Staircase 1 to be fairly safe?

Indah : In Staircase 1, if my foot size is 30 cm, then the tread and riser are both 30 cm, which would make it fairly safe because its ramp (slope) is 1.

A : What do you mean by "its ramp is 1"?

Indah : It means that I divided the riser by the tread.

A : What about the ramps in Staircases 2 and 3?

Indah : For Staircase 2, the riser is 60 cm, and the tread is 30 cm, so the ramp is 2. For Staircase 3, the tread is 60 cm, and the riser is 30 cm, so the ramp is 0.5.

A : Why did you do those calculations? What do those numbers tell you?

Indah : The ramp becomes steeper when the height (riser) is greater than the length (tread).

Activity 5: Comparing Some Points of the Road

The Comparing Some Points of the Road activity is designed to support students' understanding of slope as an algebraic ratio. In this activity, a road is represented as a connection between a series of coordinate points (x, y). Building on the earlier activity (Activity 4), where students used the concept of rise over run, we expect that students will now consider calculating the slope between two given points by examining the difference in height and the difference in distance along the x-axis. In other words, they are expected to apply the formula $\frac{(y_2 - y_1)}{(x_2 - x_1)}$. This activity is closely aligned with the mathematical formulas that students have encountered in their school curriculum. As such, students might recall and use the formula when solving this problem as shown in Figure 6.

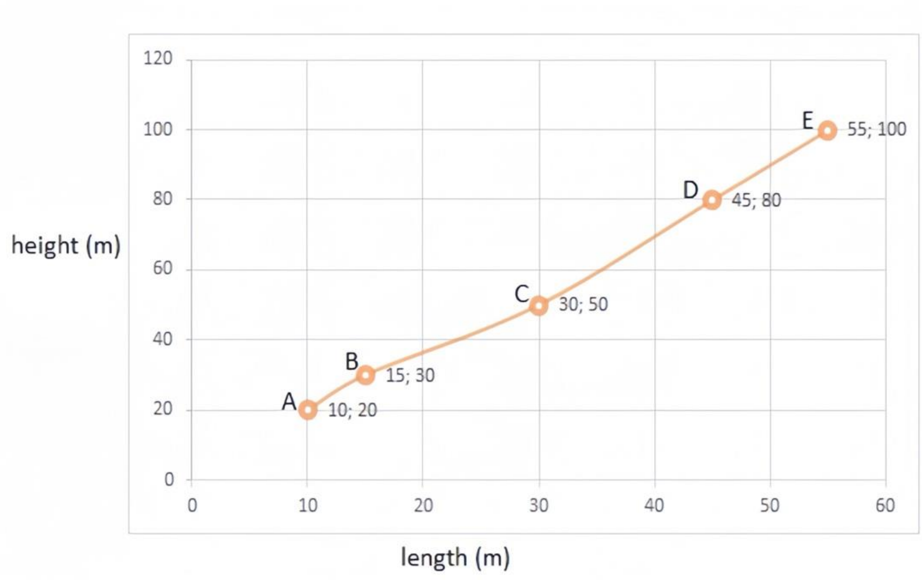


Figure 6. Some points of the road

Furthermore, an interview was conducted with the aim of exploring Indah's understanding of slope as an algebraic ratio using several points of the road.

A : What do you see in this figure?
Indah : It is a line with points A, B, C, D, and E.
A : What can you say about the slope of this figure, particularly between points A and B?
Indah : The slope is 2, Miss.
A : How did you calculate that?
Indah : $\frac{(30-20)}{(15-10)}$ *equals 2, Miss.*
A : Why did you perform that calculation?
Indah : Because that's the formula, Miss.
A : Could you compare the slope from B to C and from C to E?
Indah : The slope from B to C is $\frac{4}{3}$*, and the slope from C to E is 2.*
A : Which is safer, the slope from B to C or from C to E?
Indah : The slope from B to C is safer than from C to E because the slope is lower, Miss.

Activity 6: Comparing Top-view and Side-view of the Hill

The Comparing Top-view and Side-view of the Hill activity is designed to enhance students' understanding of slope as an algebraic ratio. In this activity, students analyze the safest route using algebraic justification. They may apply the formula $\frac{rise}{run}$, where the rise corresponds to the change in height (y-axis) and the run corresponds to the length (x-axis) of the road, to calculate the slope. This activity is closely aligned with the mathematical formulas students have encountered in their school curriculum, and as such, they are likely to recall and use these formulas when solving the problem, as illustrated in Figure 7.

Figure 7. Top-view and side-view of the hill

Furthermore, an interview was conducted to explore Indah's understanding of slope as an algebraic ratio using various top-view and side-view representations of a hill.

A : What do you think about this picture, Indah?
Indah : It looks like a hill, Miss.
A : How many hills do you see?
Indah : Two—the upper one and the lower one.
A : Are they different hills?
Indah : It's the same hill, Miss, but viewed from different perspectives.
A : What do you mean by "the same hill with different perspectives"? Please explain further.
Indah : In the top image, the hill is seen from above, and in the lower image, it's the side view of the same hill.
A : That's interesting. Can you give me a suggestion on where I should start if I want to go to the top of the hill?
Indah : It depends on the time you have, Miss. If you want to get there quickly, you can go from A to E, but if you have more time, you can go from J to F.
A : Oh, I see. So, it depends on the reason, right?
Indah : Yes.

A : What if I want to choose a safe route, and speed is not important?

Indah : Well, it might be better to take the route from J to F, since the path from A to E is steeper.

A : Why do you say that? Can you explain your reasoning?

Indah : We need to compare the slope from A to E and the slope from J to F.

A : Ah, interesting. Could you elaborate?

Indah : Sure. The slope from A to E: first, I calculate the length as $125 - 25 = 100$*, and the height as* $125 - 25 = 100$*. So, the slope is the height divided by the length:* $\frac{100}{100} = 1$*. To calculate the slope of the route from J to F: the length is* $400 - 175 = 225$ *, and the height is* $125 - 25 = 100$*. So, the slope is* $\frac{100}{225} = 0.44$*.*

A : That's great. Now we have two slope values. So, what should I choose?

Indah : You should choose J to F, Miss.

A : Why?

Indah : Because it's safer than A to E. Not only is the slope value lower, but it also looks like the line is flatter.

A : What do you mean by "flatter"?

Indah : When the slope gets lower, the road gets flatter. It looks almost like a horizontal line.

Discussion

In Activity 1, when justifying her selection of the third image, Vascha emphasized the extended descent of the road, referring to it as "down-steep." The term "steep" aligns more closely with the empirical interpretation or everyday understanding of slope (Deniz & Kabael, 2016). From a mathematical standpoint, the elevation of the descent holds greater significance than the length of the downhill path when determining the safest and most optimal route. In real-world scenarios, cyclists tend to prefer gradients that are not excessively steep due to potential safety hazards. Although Vascha's reasoning was based primarily on visual features, she exhibited inconsistency in her use of the term "steep," as she described the roads as both straight and “zig-zag”. However, according to her interpretation, linear and zig-zag configurations do not correspond to Figure 1. She suggests that a "straight" road refers to a flat or horizontal path, while "zig-zag" denotes a route that ascends or descends. In reality, “zig-zag” routes typically have multiple turns to reduce the incline of the path, in contrast to a road that continuously ascends or descends. She acknowledged that the shape of the road affects speed, using terms such as "steady" and "unsteady" to describe changes in velocity. In Figure 2, she illustrated a horizontal road where the cyclist maintains a steady speed, an uphill road where the cyclist decelerates, and a downhill road where the cyclist accelerates. Given her exclusive focus on the uphill and downhill segments, she likely lacked a comprehensive understanding of steepness.

In a subsequent activity, Indah presented a different approach. When prompted to compare the various tracks, it became clear that, while Indah observed the images, she paid closer attention to more specific details, such as the orientation of the trail. She emphasized that the direction of the starting point was crucial in determining the best route, identifying paths as flat, uphill, or downhill. She further categorized narrow mountain paths as "dangerously steep." Her response suggested that, although her reasoning remained visually focused, she demonstrated a slightly deeper understanding of the physical property of slope compared to Vascha (Deniz & Kabael, 2016). In her comparative analysis, she took into account the magnitude of the uphill and downhill segments, recognizing the need to use brakes when cycling on steeper slopes. She understood that steeper roads posed greater risks, and as the gradient became less steep, navigation became easier. Indah's comprehension of steepness as a physical property and its implications for cycling was evident. However, she demonstrated a lack of understanding of the x and y-axes, asserting that a longer uphill correlates with greater height. Mathematically, the graph representing the road is derived from the integration of both the x-axis and y-axis. Her assertion that a road with a longer uphill corresponds to greater height pertains only to the y-axis. As expected, her justification remained grounded in visual properties. While she considered aspects of the y-axis that influenced the road's characteristics, her reasoning was incomplete. Many students tend to restrict their understanding of slope to its graphical representation, focusing primarily on the line's steepness or orientation while neglecting its fundamental mathematical relationship. Research has shown that many students perceive slope primarily as a visual or physical attribute (e.g., "steep" or "flat") rather than recognizing it as a quantitative relationship between variables (Stump, 2001).

Indah exhibited a clear preference for avoiding roads with steep inclines in Activity 2, specifically Banana Road and Cherry Road, which she identified as dangerous due to their near-vertical orientation. Indah used the term "vertical line," a reference that closely aligns with the mathematical concept of the y-axis, or height, which she had previously articulated. She demonstrated the ability to interpret slope in terms of the relevant dimensions, namely the y-axis (Deniz & Kabael, 2016). Indah further asserted that a flat or horizontal line would represent a safer route, associating "horizontal" with the distance or length of the road. This suggests that she may have begun to understand steepness by linking height to the vertical line (y-axis) and length to the horizontal line (x-axis). She posited that the length of the road is determined by the distance along the x-axis and the elevation change along the y-axis. However, her understanding of the x-axis remained somewhat unclear, as she emphasized both vertical and horizontal distances when conceptualizing slope (Choy, 2006; Deniz & Kabael, 2017). Nevertheless, this represents a positive development in her understanding, as she began to recognize the importance of changes along both axes. To further facilitate her understanding, a follow-up activity was designed to direct her focus more specifically on the x-axis.

In Activity 3, the researchers manipulated the lengths of two roads—Apple Road (a) and Apple Road (b)—where the length of road (a) was shorter than that of road (b). This setup encouraged students to explore the possibility that the shorter road (a) could appear to have a greater elevation than the longer road (b). Indah responded as expected, asserting that Apple Road (a) is indeed higher than Apple Road (b). Her reasoning was coherent, as she explained that the first image appeared steeper than the second. Indah demonstrated an awareness of the x-axis in her interpretation of slope by engaging in the necessary calculations. She understood that a shorter road can seem to have a greater elevation due to its steeper gradient. When asked to compare the two roads, Indah emphasized the difference in their lengths, indicating her comprehension of the x-axis as a contributing factor to slope. This suggests that the visual representation of the components of a line is pivotal in aiding the transition toward a more mathematical understanding of slope (Deniz & Kabael, 2017). Indah performed well in focusing more on the x-axis in her analysis, demonstrating a deeper understanding of the relationship between road length and elevation. The researchers proceeded to the next activity, building on this newfound insight.

When researchers asked Indah to identify the safest staircase in Activity 4, she selected option (3) as her preferred choice. Indah considered the relationship between the height of the steps and the energy required to ascend the staircase. In her explanation, she used the concept of slope as a geometric ratio by dividing the riser (height) by the tread (length). She further recognized the impact of modifying one of the dimensions, stating that "the ramp becomes larger when the height exceeds the length." Indah conceptualized the rise as a vertical component (height) and the run as a horizontal component (length) and used these definitions to calculate the slope. This aligns with Stump's (1999) findings, which suggest that students often confuse the concepts of rise and run when discussing slope.

In Activity 5, Indah began the interview by using formal mathematical terminology, referring to the figure as a "line," although a more accurate description would be a graph that connects multiple points. Indah correctly applied the slope formula, calculating the change in y divided by the change in x, expressed as $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$. Despite her ability to compute the slope of the line, it appeared that she did not fully comprehend the concept of slope as a rate of change (Crawford & Scott, 2000). This was evident in her execution of the calculations without providing any elaborative justification or context, suggesting that she may not have fully grasped the conceptual meaning behind the calculations.

Indah demonstrated the use of formal mathematics by applying the slope formula without providing further explanation in Activity 6. Following this, we introduced a new activity to further evaluate her understanding of the algebraic ratio of slope. This exercise incorporated various representations of slope (Crawford & Scott, 2000), such as top-view and side-view perspectives, to help her grasp the concept. On the other hands, Indah also recognized that the different illustrations depicted the same concept (a hill) from distinct viewpoints. She selected a safe route and justified her choice by analyzing the path. Although her calculations aligned with formal mathematics, her reasoning was not solely based on mathematical principles. Notably, she emphasized that the road appeared "flatter," indicating her understanding that safety is not only determined by a low gradient but also by the flatness of the road surface.

CONCLUSION

For many Indonesian students, the concept of slope is often reduced to a mere numerical value, calculated using a formula that remains unclear to them. This formula is based on an algebraic ratio, and students frequently lack a physical or geometric understanding of slope. The aim of this study was to examine a series of activities designed to foster a deeper, more holistic understanding of slope, encompassing its nature as a physical property, a geometric ratio, and an algebraic ratio.

To promote a more comprehensive understanding of slope as a physical property, an activity involving bicycle race routes was developed. This exercise provided students with the opportunity to build an informal, intuitive understanding of slope, allowing them to visually justify their reasoning. The activity included both distinct and identical paths, enabling students to observe the steepness of the routes while considering factors such as height variations and horizontal changes. Additionally, comparative staircases were designed to strengthen students' grasp of geometric ratios. This approach encouraged students to evaluate the safest staircase by calculating its steepness using the rise-over-run formula.

The case of Indah illustrates how the activities facilitated the development of her understanding of slope. These activities allowed her to progress from an informal, intuitive understanding to a more formal mathematical framework. However, the final activity revealed that she reverted to using a standard formula when the context became more explicitly mathematical. In response to this, the researchers introduced a subsequent activity to address the challenges identified in the previous task. This final activity demonstrated a significant shift in students' conceptualization of slope, both informally and formally.

Such activities are critical for addressing the challenges faced by both students and educators in Indonesia regarding the teaching and learning of the concept of slope. However, it is important to note that the study does not fully represent the broader population of junior high school students, as the concept of slope remains a significant challenge in this educational context. The study's limitations, including time constraints and reliance on telephone and text-message interviews, restricted the researchers from directly observing how students engage with the concept of slope through physical interaction with their environments (e.g., staircases, slopes).

A transformation in classroom pedagogy is essential. Students should not only focus on providing correct answers but should also be encouraged to explain and justify their reasoning. The interviews with Indah further highlight the difficulties students encounter in grasping the full conceptualization of slope. Through the introduction of foundational slope concepts via diverse activities, this study emphasizes the importance of enhancing students' understanding of slope, starting with familiar, everyday contexts such as walking through the school environment, measuring various types of staircases, or exploring the properties of camping tents.

Ultimately, the study suggests that a more innovative and creative approach to teaching is necessary, allowing students to retain and apply a deeper, more flexible understanding of slope, rather than defaulting to rigid, formal mathematical interpretations. The findings call for further exploration of pedagogical strategies that integrate both informal and formal conceptualizations of mathematical concepts in order to foster more meaningful learning experiences for students.

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