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Prospective secondary mathematics teachers' use of inquiry-based teaching principles as conceptual tools when modifying mathematical tasks

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Abstract

The objective of this study was to investigate how prospective secondary mathematics teachers apply inquiry-based teaching principles to modify tasks that support students' engagement in specific mathematical practices. The research employed the theory of goal-oriented decision-making to describe and explain the use of inquiry-based teaching principles as a conceptual tool by these prospective teachers. The study involved two cohorts, comprising 43 prospective teachers (20 in one cohort and 23 in the other) enrolled in a Secondary Education Teaching program. Data were collected from written reports documenting the implementation of two professional tasks, where participants modified textbook assignments to promote exploratory teaching. An inductive analysis was conducted in two phases. The findings revealed that prospective teachers consistently applied inquiry-based teaching principles when they set specific mathematical practices as student learning objectives, such as analyzing particular cases, identifying patterns and relationships, and formulating conjectures and generalizations. However, when these mathematical practices were not established as learning objectives, teachers struggled to apply inquiry-based teaching principles consistently during task modification. These results suggest that inquiry-based teaching principles are an effective conceptual tool for prospective teachers' instructional reasoning. Nonetheless, for consistent application, it is crucial to establish a coherent network of logical connections between the conceptual tool and the intended learning objectives.

Keywords: Core Practice in Mathematics Teaching, Inquiry-Based Mathematics Teaching, Mathematical Tasks Modification, Prospective Secondary Mathematics Teachers' Learning

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Mathematics teaching is a multifaceted practice that requires the integration of knowledge, skills, relationships, and professional identity to effectively carry out specific tasks within particular educational settings (Grossman et al., 2009; Lampert, 2012). The concept of Core Practices refers to teaching strategies that are consistently implemented across various curricula and instructional methods, while maintaining the complexity and integrity of teaching (Grossman et al., 2009; Matsumoto-Royo & Ramirez-Montoya, 2021). These practices, grounded in research, enable prospective teachers to develop proficiency, and they hold significant potential for enhancing student outcomes (Grossman, 2018; Hiebert et al., 2007). Core practices encompass activities such as recognizing students' mathematical reasoning and facilitating classroom discussions (Jacobs & Spangler, 2017), alongside the design and adaptation of mathematical tasks (Kim et al., 2020; Watson & Ohtani, 2021; Sullivan et al., 2013). More specifically, mathematics teaching can be conceptualized as a cohesive set of core practices (Jacobs & Spangler, 2017; Lampert, 2012). In this context, we argue that modifying mathematical tasks is a key core practice during lesson planning, as teachers must tailor tasks to align with students' learning objectives, ensuring that the mathematical activities reflect the intended instructional outcomes (Coles & Brown, 2016; Leong et al., 2022; Choy, 2016; Riard & Kaur, 2022).

Research indicates that some teacher educators advocate for the adoption of practice-based teacher education to better align teacher preparation with core practices (Grossman et al., 2009; Hiebert et al., 2007; Zeichner, 2012). However, supporting prospective teachers in effectively implementing specific core practices in mathematics education presents considerable challenges. This may involve reshaping existing training methodologies (Kazemi & Waege, 2015; McDonald et al., 2013; Hiebert et al., 2007) and establishing a structured framework to analyze prospective teachers' learning processes (Mitchell & Marin, 2015).

Several studies have found that decision-making is a particularly challenging task when it comes to professionally noticing classroom dynamics or students' mathematical thinking. Examples of this difficulty have been documented in both international (Barnhart & van Es, 2015) and national contexts (Fernández et al., 2024; Ivars et al., 2020; Moreno et al., 2021). Specifically, in a competency-based mathematics curriculum, there is a need for further exploration of decision-making characteristics that align with the principles of inquiry-based learning.

Recent studies have explored how prospective teachers acquire the practice of task modification (Ayalon et al., 2021; Aytekin-Kazanç & Isiksal-Bostan, 2024; Lee et al., 2019; Lee et al., 2023; Paredes et al., 2020). One key learning feature identified in these studies is how prospective teachers' orientations toward the subject influence their decision-making when modifying tasks to align with specific learning goals. Here, the term orientation refers abstractly to beliefs, values, and preferences (Schoenfeld, 2015) that shape individuals' approaches to situations, guiding their decisions and use of available resources. For instance, both prospective and in-service secondary mathematics teachers addressed components of the teaching triad—mathematical challenge, learning management, and sensitivity to students (Jaworski, 1992)—differently during lesson planning, depending not only on the mathematical problem but also on their subject-specific orientations (Ayalon et al., 2021). In particular, prospective teachers' orientations regarding the nature of mathematical tasks played a significant role in their decisions on task modification. This was especially apparent when they evaluated whether certain guidelines for task modification were effective and how to utilize them to engage students in cognitively demanding tasks (Aytekin-Kazanç & Isiksal-Bostan, 2024).

Another factor influencing the learning of this core practice is the interplay between mathematical and pedagogical aspects of task modification, particularly prospective teachers' capacity to modify tasks and their understanding of the underlying mathematical and pedagogical elements (Lee et al., 2019; Lee et al., 2023). As prospective teachers deepened their grasp of design principles, they increasingly considered ways to integrate both mathematical and pedagogical features, addressing how interaction, questioning, and context could support students' learning. For example, Lee et al. (2019) found that comparing textbook tasks reflecting inquiry-based principles, along with prompts and theoretical knowledge provided by teacher educators, facilitated some prospective teachers' learning. Moreover, when prospective teachers were tasked with modifying tasks using specific tools, such as increasing the cognitive demand (Lee et al., 2023), their responses varied based on task characteristics. In particular, they were more adept at identifying tasks with lower cognitive demand than those with higher demand.

Furthermore, modifications made following the provided guidelines often failed to align with the intended learning objectives, highlighting the difficulty prospective teachers faced in adjusting tasks to meet learning goals. This finding suggests the need for further investigation into how prospective teachers understand and apply the provided tools (Smith, 2014) in relation to students' learning objectives.

These studies have demonstrated that prospective teachers often struggle to align their instructional intentions with students' mathematical thinking when modifying tasks. Specifically, it is challenging for them to grasp how the conditions for task modification interact and align with the intended learning objectives. In particular, prospective teachers find it difficult to modify tasks in ways that create opportunities for students to engage in specific mathematical practices, such as generating sets of particular cases, identifying patterns and relationships, and formulating conjectures and generalizations (Artigue & Blomhøj, 2013; Dorier & Maass, 2020). Implementing inquiry-based mathematics teaching, for instance, requires teachers to carry out a sequence of practices that enrich mathematical tasks, enabling students to generate specific cases, recognize regularities, formulate conjectures, and refine those conjectures through examination (Artigue & Blomhøj, 2013; Lee et al., 2019).

In light of these challenges, this study investigated the characteristics of prospective teachers' learning processes as they developed core teaching practices within a practice-based teacher education program. Specifically, we focused on how they learned to modify tasks to support inquiry-based teaching principles. The study's objective was to describe and explain how prospective secondary mathematics teachers acquire the ability to use inquiry-based principles, framed as a conceptual tool (Smith, 2014), to effectively modify mathematical tasks.

We employed two theoretical frameworks to investigate how prospective teachers learn the core practice of task modification: Schoenfeld's Resources-Orientations-Goals (ROG) model of teacher decision-making (Schoenfeld, 2010) and inquiry-based teaching principles in mathematics education as a conceptual tool to guide prospective teachers' reasoning (Artigue & Blomhøj, 2013; Dorier & Maass, 2020).

From the perspective of the ROG framework, mathematical tasks are integral to fostering students' mathematical understanding. Depending on the characteristics of a task, students may either follow routine procedures or be challenged to explore connections between mathematical concepts and properties, particularly when solving non-routine problems (Artigue & Blomhøj, 2013; Stein & Smith, 2011). Therefore, modifying tasks to enhance students' mathematical thinking is a critical problem in mathematics teaching (Schoenfeld, 2010; 2011). We adapted the ROG framework to analyze the decision-making processes of prospective secondary mathematics teachers as they modified tasks using inquiry-based teaching principles. Our premise was that we could describe and explain their decision-making by examining the relationships between their knowledge, orientations, and goals. In this context, the ROG framework was suitable for understanding task modification as a goal-directed activity underpinned by specific knowledge (Chin et al., 2022; Riard & Kaur, 2022), making it an effective lens for analyzing prospective teachers' learning of this core practice.

Schoenfeld (2010) posits that teachers' decision-making in learning situations is influenced not only by their knowledge, goals, and educational context but also by their beliefs and values. The ROG framework (Resources, Orientations, Goals) provides a structure for understanding how these elements interact to shape teachers' reasoning and decision-making in instructional settings. Each component of this framework is discussed.

"Resources" (R) encompass the broad knowledge base that prospective teachers utilize, including procedural knowledge (how to perform tasks), conceptual understanding (why things function as they do), and heuristics or problem-solving strategies. In the context of this study, resources refer to the knowledge acquired during teacher education programs, such as inquiry-based teaching principles used by prospective teachers to modify mathematical tasks and create opportunities for developing specific mathematical practices (Artigue & Blomhøj, 2013; Barquero & Jenssen, 2020; Dorier & Maass, 2020; Smith, 2014).

"Goals" (G), either explicit or implicit, are the intended learning outcomes that prospective teachers aim to achieve through their instructional decisions. For instance, in the framework of inquiry-based teaching, goals might involve guiding students to identify patterns in specific cases to derive conclusions about a mathematical situation (Towers, 2010). These learning goals are pivotal in directing the prospective teachers' decisions when modifying tasks.

Lastly, "Orientations" (O) refer to the teachers' beliefs, values, and preferences, which interact with resources and goals to influence decision-making in teaching. For example, a teacher who emphasizes procedural fluency over conceptual understanding will likely prioritize different aspects of task modification compared to one who values deep conceptual learning (Schoenfeld, 2011). These orientations help explain variations in how teachers modify tasks and approach instructional planning.

While the ROG-framework was originally developed to explain individual decision-making processes, we adapted it in this study to examine the collaborative work of groups of prospective secondary mathematics teachers. We assumed that when a group collectively modified and justified a mathematical task, they were taking into account both the learning goals and the underlying teaching principles. On the other hand, when a group failed to align their modifications with the learning objectives, it indicated a lack of consideration of these principles. To analyze this collective work, we focused on the interplay between resources, orientations, and goals within the group context.

The second theoretical perspective guiding our study involves inquiry-based teaching principles, which prospective mathematics teachers used as a tool to modify tasks. Inquiry-based mathematics teaching is a student-centered model designed to mirror the work of mathematicians, as described by Dorier and Maass (2020). In this approach, students are encouraged to observe phenomena, ask questions, and develop mathematical explanations to address those questions. This method involves interpreting and evaluating solutions, as well as communicating and discussing the outcomes effectively. Mathematical tasks adhering to inquiry-based principles allow students to explore specific cases, establish conjectures, make generalizations, and evaluate their solutions (Lee et al., 2019).

By focusing on tasks that embody these characteristics, teachers create opportunities for students to build connections between mathematical concepts. Sequences of well-structured questions guide students toward achieving learning goals, such as developing mathematical modeling competence (Greefrath, 2020). Through exploration and systematization, students engage in processes like conjecturing, proving, and using counterexamples to deepen their understanding. Artigue and Blomhøj (2013) highlight that inquiry-based teaching fosters a dialectical interaction between proof and refutation, leading to the development of solid mathematical reasoning and generalization. Teachers play a crucial role in facilitating this process by guiding students through progressively challenging tasks that encourage exploration, reasoning, and communication of mathematical ideas.

From this perspective, inquiry-based teaching principles determine how mathematical tasks should be designed (Barquero & Jenssen, 2020; Towers, 2010), and we can consider them as the resources (this is, a tool to be used) which prospective teachers employ to learn to modify mathematical tasks. In this sense, a mathematical task that supports inquiry-based learning would allow for different solutions and help to shape students' argumentation. Furthermore, the tasks should enable studying particular cases, looking for patterns, or searching for counterexamples to make conjectures and to verify them in order to generalize results or establish conclusions.

Using inquiry-based principles to organize teaching implies recognizing specific mathematical practices as teaching targets. Prospective teachers must conceive school mathematics as a practice, not only as a set of concepts and procedures. We consider that task modification constitutes a relevant, goal-oriented behavior in a context of curricular reform which explicitly encourages the implementation of inquiry-based principles in teaching and learning scenarios.

The present study aimed at characterizing prospective secondary mathematics teachers' use of inquiry-based teaching principles as a tool through which to approach task-modification to support students' specific mathematical practices. To achieve this goal, we posed the following research questions:

1. How do prospective secondary mathematics teachers modify tasks to support inquiry-based teaching?
2. To what extent are their decisions consistent with inquiry-based principles and student learning goals?

METHODS

The study involved two cohorts comprising 43 prospective secondary mathematics teachers (20 in the first cohort and 23 in the second), all enrolled in a Secondary Education Teaching program at a Spanish university. These participants provided informed consent to take part in the research project. The postgraduate program, spanning two semesters, is accessible to prospective teachers from various disciplines, including mathematics, physics, and engineering, and equips them to become qualified secondary mathematics educators.

The curriculum of the program integrates pedagogical and psychosocial subjects alongside specialized courses in mathematics education. It also includes a four-week field experience in a secondary school, allowing prospective teachers to apply their learning in a practical setting. The mathematics education courses are designed to impart essential resources for mastering core mathematics teaching practices. These practices encompass recognizing and responding to students' mathematical thinking, modifying mathematical tasks to promote inquiry-based learning, planning mathematics lessons that incorporate technological tools, and critically evaluating curricular materials, such as mathematical textbooks and digital resources.

Data for the study were gathered through the participation of prospective teachers in the Mathematics Teaching course, which lasted one semester. The course objectives included learning to analyze, modify, and sequence mathematical tasks, planning lessons that support inquiry-based learning, and utilizing GeoGebra as a technological resource. To facilitate their learning, prospective teachers were provided with information regarding the secondary mathematics curriculum and the principles of inquiry-based teaching, both of which are aimed at enhancing specific mathematical practices among students. The Spanish secondary mathematics curriculum is competency-based and emphasizes inquiry-based teaching, recommending that particular mathematical practices be considered as learning objectives, such as generating specific cases, and exploring the design and implementation of tasks and activities within the classroom context (Artigue & Blomhøj, 2013).

The Mathematics Teaching course was facilitated by an instructor who was also one of the researchers, responsible for monitoring the collaborative work among the prospective teacher groups.

This instructor addressed questions that arose during the group activities and curated the various ideas presented to prepare for subsequent discussions.

As part of the task-modification module, the prospective teachers were assigned two professional tasks (illustrated in Figures 1 and 2) to be addressed collaboratively in groups. Each cohort consisted of five groups, with each group comprising three to five prospective teachers. Over a period of three weeks, these groups were tasked with solving the assigned problems, utilizing both face-to-face (in three 2-hour sessions) and non-face-to-face modalities.

The primary data for this study were derived from the final reports submitted by each prospective teachers' group, in which they described and justified their decisions regarding the solutions to the two professional tasks. These reports served as a key resource for analyzing how prospective teachers modified tasks to align with inquiry-based teaching principles and support specific mathematical practices.

The research instrument consisted of two professional tasks, A and B, designed to assess the abilities of prospective teachers. These tasks included a textbook problem and six professional questions. Both tasks were developed by the researchers and provided by the instructor. Task A involved a Grade 7 textbook problem, referred to as Problem A, which focused on the explicit learning goal of exploring the relationship between the area and perimeter of polygons (Figure 1). The prospective teachers were required to respond to six questions as part of their professional training:

1. Identify the content, learning goals, and indicators based on the curriculum.
2. Solve the problem.
3. Analyze the concepts, procedures, and properties necessary to solve the problem.
4. Evaluate the strengths and limitations of the problem in achieving the stated learning objectives.
5. Modify the problem according to the insights from question 4.
6. Design a sequence of activities that would help students achieve the learning objectives.

More specifically, questions 1, 2, and 3 addressed how the problem could be integrated into the official curriculum, the solution to the problem, and the mathematical knowledge—such as concepts, procedures, and properties—that students need to engage with at this level to solve it. Question 4 asked prospective teachers to assess the potential and limitations of the problem in supporting inquiry-based learning, considering the students' learning goals. Finally, questions 5 and 6 required the prospective teachers to modify the problem to further support inquiry-based teaching and to propose a sequence of activities incorporating the modified problem, aimed at facilitating students' understanding of the relationship between polygon area and perimeter through specific mathematical practices.

In this professional task, prospective teachers were required to address two essential aspects. First, a defined learning goal was provided. Second, they needed to understand the nature of the associated mathematical concept, specifically the fact that no direct relationship exists between the area and perimeter of plane figures. This concept is crucial for secondary students to grasp, as the relationship between a shape's area and perimeter can vary depending on the shape. In other words, the area and perimeter of a shape are independent variables; changes in one do not necessarily affect the other (D'Amore & Fandiño, 2007; Stone, 1994). Within this framework, inquiry-based teaching should help students comprehend counterexamples, thereby fostering a second learning objective related to the development of mathematical practices associated with inquiry-based learning. This approach emphasizes the significance of exploration, gradual systematization, and the use of counterexamples in constructing mathematical arguments.

To initiate students aged 12 to 13 years to the relation between area and perimeter, a secondary teacher selects the following task:
41. On a grid, draw five different figures that can be formed with five squares. They are called pentaminoes.
Define the perimeter of each figure
Do all of them have the same area?

Figure 1. Problem A from the textbook

Professional Task B involved a textbook problem, referred to as Problem B, designed for secondary students aged 12 to 13 years (Figure 2). It included the same set of questions as Professional Task A. In Task B, prospective teachers were required to consider two key elements. First, they needed to define a specific learning goal. Second, they had to address several important mathematical facts: two figures are considered similar if their corresponding angles are congruent, and the ratios of the lengths of their corresponding sides are equal. The similarity factor (k) represents the ratio between the lengths of the sides in two similar figures, while the relationship between the areas of two similar figures is expressed as k^2.

In this context, inquiry-based teaching aimed to explore the role of progressive systematization and pattern recognition through specific examples. By identifying patterns, students could make conjectures and ultimately prove them, fostering a level of certainty that no further counterexamples would invalidate the conclusion. This process creates opportunities for generalization, aligning the task with the goals of inquiry-based teaching.

We wish to reproduce the following figure on a 3/2 scale.
a) Please draw the extended figure.
b) Calculate the length of the sides

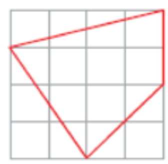


Figure 2. Problem B from the textbook

Professional Tasks A and B posed the same set of questions to prospective teachers, with the goal of assessing how they utilized inquiry-based teaching principles, established students' learning goals, and the decision-making guidelines they followed. The questions required prospective teachers to analyze the problems in relation to specific curricular levels, identify the possibilities and limitations of using the tasks to support inquiry-based learning, modify the problem accordingly, and incorporate it into an activity sequence designed to foster specific mathematical practices in students. These practices could include tasks such as organizing data, identifying patterns, making conjectures, and generalizing results.

Despite the shared structure, there were key differences between the two tasks. In Professional Task A, the learning goal was explicitly provided, while in Professional Task B, prospective teachers were required to define the learning goal themselves. This distinction enabled a closer examination of how the presence of an explicit learning goal influenced the teachers' approach to modifying the problem and creating an activity sequence grounded in inquiry-based teaching. Specifically, the prospective teachers' reasoning and justifications provided insight into their instructional decisions, revealing how these were shaped by the instructional goals they set, the resources they utilized, and the teaching orientations they adopted in the given context (Schoenfeld, 2010).

In analyzing the data, we examined 20 reports written by 10 groups of prospective teachers, with each group submitting two essays in response to both professional tasks. The analysis was conducted using the ROG framework in conjunction with the principles of inquiry-based teaching, following an inductive process as outlined by Corbin and Strauss (2008). Initially, three researchers independently reviewed the reports, identifying excerpts that reflected key elements such as the identification or formulation of the learning goal, the possibilities and limitations of the problems, and the characteristics of the modified problems. These excerpts were categorized as preparatory comments (Table 1).

Table 1. Sample of preparatory comments

Fragment of the report on task A Group 3	Preparatory Comments
3. Analyze the mathematics required to solve this task: concepts, procedures and/or properties. Calculation of area and perimeter of polygons. Relationship between both. Decomposition into simple figures. The aim of the task is that pupils observe how figures with the same area can present different perimeters.	Learning goal given is recognized
4. Analyze the possibilities and limitations of this task with respect to achieving the proposed goals. One limitation is that because the question is open, it is very easy to draw all the figures with the same perimeter and establish erroneous conclusions, e.g., that the same perimeter corresponds to the same area. The question does not ask for any justification, so students can simply answer randomly or without really understanding the concept.	They mention the difficulties in fulfilling the objective in terms of establishing erroneous conclusions. Types of questions that do not allow reasoning.
5. Modify the task according to the answer to section 4. On a grid, draw different figures in such a way as to meet the following criteria: a. Three figures must have the same area, but different perimeters. b. Two figures must have the same perimeter, but different areas. c. Draw a figure with 5 squares and the scaled figure with a ratio of 1:2. How do the perimeter and area increase? d. Do you think there is a relationship between area and perimeter?	-Particular cases: -Search for counterexamples -Different ways of solving the problem. -Conjecturing and checking conjectures. -Connections between properties

Table 1 provides a sample of these preparatory comments, drawn from Group 3's report on Task

A. For example, excerpt 3 focuses on analyzing the mathematical concepts, procedures, and properties required to solve the task, including the calculation of the area and perimeter of polygons, the relationship between them, and the decomposition of complex figures into simpler ones. The researchers noted that the group successfully recognized the learning goal, as reflected in the comment, "The learning goal provided is recognized."

Similarly, excerpt 4 explores the possibilities and limitations of the task in achieving the proposed learning objectives. Group 3 identified a limitation in the open-ended nature of the problem, where students might mistakenly conclude that figures with the same perimeter must also have the same area. The researchers highlighted this insight by commenting on the potential for students to reach erroneous conclusions and the types of questions that might inhibit deeper reasoning.

Excerpt 5 illustrates how the task was modified by Group 3 in response to these limitations. The modified task asked students to draw figures under specific conditions (e.g., figures with the same area but different perimeters and vice versa) and to explore the relationship between area and perimeter. The researchers noted that this modification encouraged the use of inquiry-based teaching strategies, such as searching for counterexamples, exploring different problem-solving methods, making conjectures, and checking them, as well as fostering connections between mathematical properties.

In the second phase of the analysis, we compared the individual descriptions of each professional task to identify similarities and differences, focusing on three emerging themes: (1) learning goal identification, which relates to the target teaching objectives; (2) possibilities and limitations of the problem, particularly concerning the application of inquiry-based teaching principles; and (3) features of task modification and activity sequence design for inquiry-based instruction. We examined these themes to explore how prospective teachers employed inquiry-based teaching principles as resources when modifying the problems, evaluating whether their application of these principles remained consistent across both tasks.

We also analyzed the coherence of the groups' decision-making processes, considering both the specific issues raised by each professional task and the consistency of their responses between Task A (where the learning goal was explicit) and Task B (where it was not). This approach allowed us to assess how the prospective teachers' orientations, beliefs, and use of inquiry-based principles influenced their decisions and whether their reasoning aligned logically with the principles of inquiry-based teaching. We particularly looked for any contradictions in their application of these principles between the two tasks.

Table 2 outlines the themes along with the corresponding focus areas and guiding questions used to characterize each theme. These themes emerged from the interplay between the elements of the theoretical frameworks applied in this study. For each problem, we identified the learning objectives, evaluated whether the principles of inquiry-based teaching were utilized as a resource to analyze the problem's possibilities and limitations, and assessed whether the decision-making process and the modified tasks were aligned with both the established learning goals and the principles of inquiry-based instruction.

To ensure the reliability of our coding, whenever there were disagreements in our interpretations of the data, we revisited the original dataset to seek out dissenting or corroborating cases, triangulating our conclusions. Using this analytical approach, we identified two key features that characterized how prospective secondary mathematics teachers modified tasks to align with inquiry-based teaching principles. These features, along with their implications, are detailed in the Results section.

Table 2 outlines the questions pertinent to the themes that guided the analysis. For example, with respect to the theme of employing inquiry teaching principles to evaluate the strengths and limitations of the problems, we sought evidence from the reports of prospective teachers concerning how they justified the extent to which the problems facilitated the attainment of educational goals. Furthermore, we assessed whether they took into account the principles of inquiry teaching when evaluating the appropriateness of the problems.

Table 2. Generated themes describing and explaining how prospective teachers modified the problems

Theme	Focus and questions: How are resources, goals, and orientation interconnected when prospective teachers modify the problem?
Goal identification	To what extent did prospective teachers consider the learning objective in Problem A or formulated and took it into account in Problem B?
How inquiry-based teaching principles are used as resources to analyze the possibilities and limitations of the problems	How did prospective teachers justify the extent to which the problem allowed them to achieve the learning goal? How did prospective teachers take into account the characteristics of inquiry-based teaching principles as resources to determine the suitability of the problem for inquiry teaching?
How were inquiry-based teaching principles used to decide the teaching sequence with respect to the learning goal?	To what extent did the modifications allow generating learning opportunities for students to achieve the learning goal? How did the modifications support inquiry-based teaching?

RESULTS AND DISCUSSION

The findings of this study identified two primary characteristics that delineate the manner in which prospective secondary mathematics teachers employ inquiry-based teaching principles when adapting mathematical tasks. This adaptation is influenced by the interaction of their available resources—specifically their understanding of inquiry-based principles—their pedagogical orientations, and the establishment of learning objectives for their students.

The first characteristic pertains to the consistent application of inquiry-based teaching principles in task modification, whereby the prospective teachers consider the students' learning goals and align these with the inquiry-based approaches (G1, G2, G3, G5, G6, G7, G10). Conversely, the second characteristic arises when prospective teachers fail to apply inquiry-based teaching principles consistently during task modification, neglecting to explicitly consider the learning objectives (G4, G8, G9). The interplay between the use of inquiry-based principles and the definition of students' learning goals is influenced by prospective teachers' orientations, specifically their beliefs regarding student learning in mathematics and the nature of secondary school mathematics content.

To elucidate each characteristic, we will present excerpts from the reports submitted by the groups. For the first characteristic, we will provide evidence from Group 2's response to Problem A and Group 5's response to Problem B. To illustrate the second characteristic, we will reference the responses of Group 4 to both Problems A and B.

Regarding the first characteristic, which involves aligning learning objectives with inquiry-based principles, it is imperative for prospective secondary mathematics teachers to recognize the diverse types of reasoning required for solving both problems. When this alignment is effectively achieved, the specific mathematical practices essential for addressing each problem become the focal point of the decisions made in task modification, thereby justifying the sequence of problems designed to meet the learning goals. In this context, the distinct learning objectives associated with Problems A and B were articulated, and inquiry-based principles were explicitly established as resources to facilitate the specific mathematical practices required for each problem.

At this initial stage of decision-making, prospective teachers acknowledged both the potential advantages and limitations of the original problem in relation to the diverse mathematical practices involved, such as constructing particular cases, organizing information, and identifying patterns in the data. Consequently, their instructional decision-making remained aligned with the proposed goals. The decisions implemented enabled the modification of the problems to enhance reasoning: in the case of Problem A, prospective teachers sought counterexamples to the mathematical assertion concerning the relationship between perimeter and area, whereas, for Problem B, deductive reasoning was employed, leveraging the analysis of specific cases and the identification of patterns to facilitate generalization during the problem modification process.

One notable example of task modification was presented by Group 2 in their response to Problem A (see Figure 3). The objective of this problem was to explicitly demonstrate that no correlation exists between areas and perimeters; specifically, that the statement "same area implies same perimeter" is false, which is achieved through the construction of a set of specific cases. To meet this objective, the group proposed a series of questions aimed at investigating the area-perimeter relationship using three tetrominoes (1a, 1b, and 1c), while also searching for a figure that does not satisfy the conditions of that relationship, thereby identifying a counterexample. They suggested that students seek a pentomino exhibiting a distinct area-perimeter relationship (Question 2a) and examine the circumstances under which this relationship holds true or false (Question 2b). This modification is anticipated to guide students toward accomplishing the learning objectives while fostering specific mathematical practices necessary to understand this mathematical concept.

1. Please draw three different figures formed by four squares. Each square must have at least one edge coinciding with the adjacent square. Calculate the perimeter and area of each figure.
   a. Is the area equal in all cases? Why?
   b. Is the perimeter the same in all cases? Why?
   c. Is there any relationship between area and perimeter? If so, can you find one?

2. In the case of pentominoes, i.e., figures that can be formed with 5 squares with at least one edge coinciding with the adjacent square, there are 12 possible solutions, and there is an absence of relationship between area and perimeter in only one of them.
   a. Can you find it?
   b. When is the condition that two figures with the same area have the same perimeter not fulfilled?

Figure 3. Modification of problem A (Figure 1) – Group 2

In examining the possibilities and limitations of Problems A and B, it is posited that when learning goals are effectively aligned with inquiry-based principles, the problems should enable students to formulate arguments, identify connections through the analysis of specific cases, search for counterexamples, and draw informed conclusions. For instance, Group 2 articulated the possibilities and limitations associated with Problem A:

> *[Students] have the possibility of working on the area and perimeter concepts [in Problem A]. However, upon solving it, we identified several limitations:*
>
> - *There are 12 potential pentominoes, of which only one possesses a distinct perimeter. Given that students are tasked with creating only five figures, it is possible that all selected figures will ultimately share the same perimeter. Since these figures also have identical areas, students may incorrectly infer that figures with the same area will also have the same perimeter, which is false. In other words, they could arrive at an erroneous generalization.*
> - *Students may lack familiarity with pentominoes. Since the problem merely specifies that the figure consists of five squares, they might inadvertently construct the yellow figure. It is essential to clarify that the squares must share at least one edge.*
> - *In part (b), the question can be answered with a simple 'yes' or 'no,' without the need for justification. Consequently, we will be unable to ascertain whether the students comprehended the problem.*
>
> *For all these reasons, we believe that the problem is inadequate to achieve the proposed goal."*

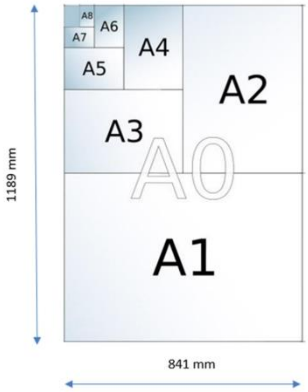
This response underscores that limiting students to drawing only five pentominoes (the analysis of specific cases) could lead to the selection of figures that share both the same area and perimeter, thereby promoting a misleading relationship between the two. Furthermore, the prospective teachers emphasized that questions should not be answerable solely with a 'yes' or 'no' response, as this does not facilitate meaningful argumentation.

When learning objectives are aligned with inquiry-based teaching principles, the focus of task modification explicitly acknowledges the necessity of cultivating specific mathematical practices, such as data sorting to identify patterns. For instance, another group of prospective teachers (G5) proposed modifications to Problem B (see Figure 4), with an emphasis on ratio, proportion, and scaling. Their aim was to explore the relationships between the areas and side lengths of similar rectangles by comparing the dimensions and areas of various DIN formats (A0 through A7), representing specific cases, and organizing the resulting data in a table.

In this modification, the group of prospective teachers formulated a sequence of questions designed to investigate the connections between consecutive formats (Questions a and b), thereby enabling students to recognize the proportional relationships among different DIN sizes and facilitating generalization. To support this inquiry, they proposed a hands-on activity (Question c) that would allow students to formulate conjectures and examine the ratios (Questions d and e), focusing on either side lengths or areas.

The decisions made exhibited a common characteristic: they thoroughly reflected on the significance of the inquiry-based principles that informed the specific learning objectives. These responses acknowledged the limitations of the original task in relation to the desired learning outcomes established by inquiry-based teaching principles, thereby unveiling a network of logical connections between the meanings of these principles and the intended learning goals. Specifically, the decisions took into account particular mathematical practices, such as generating a data set and systematically organizing that data, to create learning opportunities for formulating conjectures and supporting argumentation.

Sheet A0 has a rectangular shape and an area of 1 m2. To obtain the next format, A1, A0 is cut at the midpoints of the larger side. If we repeat the procedure with an A1, we obtain an A2, and so on.



Format	Dimension (mm)	Area (mm^2)
A0	1189 x 841	1×10^6
A1		
A3		
A4		

a) Complete the table.
b) What is the relationship between the area of one size and the area of the next size? Why? Justify your answer.
c) Take two sheets of A4 paper (two sheets of paper from your notebook) and fold one along one of its diagonals and mark this line. Fold the other one in half on the long side and cut it in two, as shown in the figure. This will give you two A5 sheets of paper. Take one of them and draw a diagonal. Overlap the A4 sheet and the A5 sheet at one vertex so that their sides coincide. Do the diagonals also coincide? What does this mean? Use a drawing to describe your answer.
d) Repeat the procedure until you obtain an A6, an A7, and an A8. Next, obtain an A3 and superimpose all the formats. What do all the DIN A rectangles look like? Justify your answer.
e) What is the similarity ratio if all sizes have the same shape?
e.1.) Look at the relationship between the areas of two consecutive sizes. Can you predict the ratio of similarity between one size and the next? Why? Justify your answer with a reasoned explanation.
e.2.) Prove it with an operation. Call "a" and "b" the dimensions of the large rectangle. What are the dimensions of the small rectangle? How long is the largest side, and how long is the smallest side? Write down the measurements of the small one and establish the proportion between the sides. Represent your answer.
e.3.) Work with this proportion and find the ratio value between the sides of an A4 paper sheet. What is it? Does it coincide with your prediction? Justify your answer.

Figure 4. Modification of problem B – Group 5

We will now address the second characteristic, which pertains to the failure to recognize learning objectives that align with inquiry-based teaching principles. This trait was identified when the distinct types of reasoning necessary to solve both problems did not facilitate the establishment of learning goals. Consequently, the decisions regarding task modification and the integration of inquiry-based principles were not congruent with the intended learning outcomes. In this instance, the prospective teachers' choices did not adequately assess or interpret the possibilities and limitations of the problems to consistently support inquiry-based teaching.

For example, the modifications suggested by Group 4 for Problem A focused on calculating the areas and perimeters of various figures as the learning objective: “Concept and calculation of the perimeter of different flat geometric figures. Concept and calculation of areas by decomposition into simple figures” (to generate a specific data set). However, this modification alone fails to address the reasoning required by the problem (which concerns the absence of a relationship between area and perimeter) and does not incorporate the essential characteristics of inquiry-based teaching (see Figure 5).

Modified problem
The following exercise is proposed:
a) Obtain the perimeter of each figure.
b) Obtain the area of each figure. You can use decomposition into simpler figures.

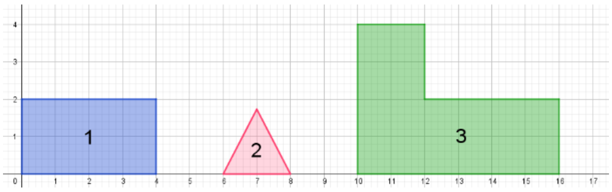


Figure 5. Modified Problem A – Group 4

In this modification, the prospective teachers’ decision to align inquiry-based principles with learning goals did not adequately consider the limitations of the task. This group believed that requiring students to draw various pentominoes presented an opportunity to foster creativity; however, the intended learning objective focused on calculating areas and perimeters. They failed to recognize this as an opportunity to highlight the absence of a relationship of the form “same area implies the same perimeter.”

Regarding the limitations of the problem, they noted that the types of figures required for the task were quite restrictive. This limitation hindered the exploration of other types of polygons. Nevertheless, they did not address how these problem limitations impacted the inquiry-based teaching approach, stating:

> *Possibilities: the problem asks them to calculate perimeters and areas of 5-plane geometric figures […]. In this way, since students have to define the geometrical figures, the exercise focuses on creativity as well as geometric and spatial vision.*
> *Limitations: because they are given the grid, they can only draw rectangular figures with parallel or perpendicular sides, so no other geometric figures (such as triangles and quadrilaterals or pentagons) are worked on.*
> Consequently, they proposed a didactic sequence in which the teacher is placed at the center of *the* teaching and learning process and in which students are expected to master the procedures, as illustrated by the proposed sequence in relation to problem A:
> *Part 1: Explanation of the perimeter concept*
> *The teacher explains the definition of the perimeter, giving different examples of perimeter calculations and using the GeoGebra tool so that students understand the reason for the different and more complex formulas to calculate perimeters.*
> *Students are asked to calculate the perimeters of different geometric shapes so that they can ask any questions they may have when carrying out these problems.*

Part 2: Explanation of the concept of area
The teacher explains the definition of the area, using GeoGebra to help students understand why they are calculating the area of each figure.
Students are asked to calculate the areas of different geometric shapes to check whether they have understood the procedure and to ask any questions they may have.
Part 3: Resolution of the modified problem so that they can apply both concepts together and ask any questions.

This teaching sequence delineated the procedures that prospective teachers deemed essential for solving the modified Problem A. Initially, the teacher introduced the concept of perimeter and its calculation, utilizing dynamic environments such as GeoGebra to provide illustrative examples. Following this, the teacher articulated the intention behind the proposed activities aimed at practicing the procedure, albeit without detailing the specific activities (Part 1). Subsequently, a similar approach was suggested for calculating areas (Part 2). Ultimately, they concluded that students were now equipped to tackle the modified problem (Part 3) as they possessed the requisite mathematical procedures.

In the context of the modification for Problem B (see Figure 6), this group of prospective teachers incorporated a question related to calculating the areas of similar figures, as the original problem focused solely on determining the lengths of the sides. Their intent was for students to delve deeper into the procedural elements, stating as a learning objective: "To calculate lengths of sides and area," while neglecting the characteristics of inquiry-based learning. Their focus remained on calculating the lengths of the sides of similar figures and the application of the Pythagorean theorem, both of which are necessary procedural components for solving the problem. They observed: "Possibilities: the task allows us to concentrate on the concept of scale and to examine it both graphically and numerically. It also facilitates students' visualization of the application of the Pythagorean theorem in calculating each side." However, their acknowledgment of limitations was limited to the assertion that the problem did not permit area calculations using general expressions. Once again, they failed to address any inquiry-based teaching considerations, stating: "- Limitations: the task does not include calculating areas."

Modification of Professional Task B:
We wish to reproduce the following figure on a 3/2 scale.

a. Please draw the extended figure.
b. Calculate the length of the sides
c. Calculate the area of the initial figure and give the area of the scaled figure.

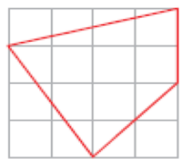
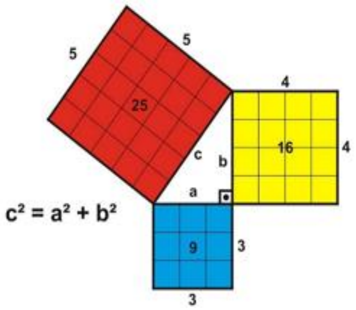


Figure 6. Modified Problem B - Group 4

This group of prospective teachers suggested a similar approach for the modified Problem B (refer to Figure 7). Initially, the teacher provided an explanation of the Pythagorean Theorem, after which students engaged in practice exercises related to it (Part 1). Following this, the same instructional sequence was applied to explore the relationships between the areas of similar figures and the lengths of their sides. However, this sequence did not facilitate an opportunity for students to independently discover these relationships (Part 2). The sequence concluded with the teacher instructing the students to solve the modified problem (Part 3).

Part 1: Teaching the Pythagorean theorem:

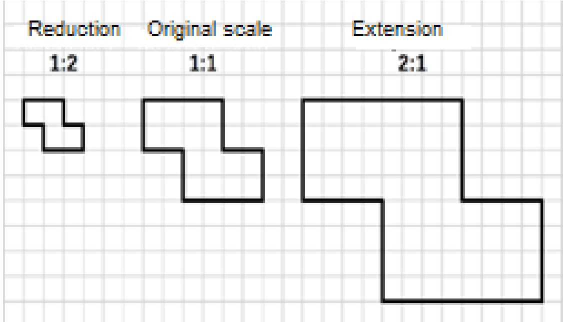
- The teacher introduces the Pythagorean theorem by reciting the definition ad nauseam to ensure that students memorise it in the long run and can always refer to this definition. The teacher emphasises that the theorem is only valid for right-angled triangles.



- The teacher presents the Pythagorean theorem graphically, using the following visual support:
- The teacher asks them to solve some problems as a group and solve their doubts.

Part 2: Teaching the concept of scale.

- The teacher presents the concept of scale based on the following example with grids so that students can count the squares and see the relationship between the figures according to their scale.



- The teacher presents some problems for the students to solve as a group and to solve their doubts.

Part 3:

- Resolution of the task proposed in section 5 of the practice so they can apply both concepts together and solve the questions that may arise.

Figure 7. Teaching sequence for Problem B- Group 4

The responses from the groups of prospective teachers exhibiting this characteristic indicated a limited and fragmented understanding of inquiry-based principles and the associated learning objectives. While some groups concentrated on certain principles, such as the generation of specific data sets, they neglected others, such as the necessity for systematic data organization. This lack of attention highlights the absence of a coherent logical framework connecting the principles to the intended learning goals.

In examining the role of inquiry-based teaching principles as a framework for evaluating and modifying mathematical tasks, it appears that effective task modification grounded in these principles necessitates the formulation of interconnected and challenging questions that engage students' mathematical practices (Barquero & Jenssen, 2020; Dorier & Maass, 2020; Towers, 2010). This perspective emphasizes the importance of viewing inquiry-based principles as an integrated set rather than isolated components. Nonetheless, the process of modifying tasks based on inquiry principles seems to be hindered by challenges in establishing a comprehensive network of meanings that encompasses both the principles and the targeted educational outcomes.

When prospective teachers design sequences of questions and problems that embody inquiry-based principles, they tend to adopt perspectives that view school mathematics as interconnected structures. This indicates an understanding that mathematics teaching and learning should be student-centered, with an emphasis on fostering conceptual understanding (Riard & Kaur, 2022). In such instances, these teachers effectively utilize available resources to create learning opportunities that engage students in tasks requiring conjecture-making, reasoning explanation, claim validation, and argumentation regarding mathematical validity (Artigue & Blomhoj, 2013). This approach is successful when prospective teachers comprehensively address and integrate the relationships among inquiry-based principles and specific mathematical practices, framing them as learning goals for students. As a result, they can modify tasks consistently with these principles.

Conversely, when prospective teachers fail to recognize specific mathematical practices as learning objectives, their task modifications become inconsistent with the resources offered in the postgraduate program. In these situations, the learning goals diverge from the original objectives or lead to a learning experience that is disconnected from the inquiry-based perspective. The orientations displayed by these prospective teachers appeared to be rule-based (Riard & Kaur, 2022), emphasizing procedural fluency and positioning the teacher at the center of the instructional process, aligning with a content-focused conception of mathematics education.

Our analysis of the role of prospective teachers' orientations in employing inquiry-based teaching principles complements findings from prior studies (Ayalon et al., 2021). Specifically, our results highlight the potential influence of these orientations on the learning of inquiry-based principles as a means to support the modification of mathematical tasks. Some prospective teachers in this study successfully modified tasks in accordance with inquiry-based teaching principles, thereby enhancing student opportunities to formulate arguments in support of mathematical claims or to recognize the absence of patterns within a data set. However, for some, it proved challenging to propose goal-directed lessons aligned with inquiry-based teaching principles. Thus, consistent with research on teacher practice, it may be beneficial to present preservice teachers with scenarios that involve conflicting objectives, encouraging them to reflect on their strategies for managing such situations (Thomas & Yoon, 2014).

CONCLUSIONS

In this study, we examined the application of inquiry-based teaching principles by prospective secondary mathematics teachers as they learned to modify tasks aimed at enhancing specific mathematical practices among students. Through this investigation, we identified key characteristics of the learning process involved. We provided the prospective teachers with a set of inquiry-based teaching principles, as delineated by Smith (2014), to assist them in defining learning objectives that align with these principles and in adapting tasks accordingly. Additionally, we employed Schoenfeld's ROG framework (resources, orientations, and goals) to elucidate how prospective teachers engaged in the task modification process, which is integral to mathematics instruction. The results indicate that inquiry-based teaching principles can serve as effective tools for supporting prospective teachers' reasoning concerning mathematical tasks; however, consistent application of these principles necessitates that prospective teachers establish a network of logical connections between the principles' meanings and the intended learning objectives. Our findings suggest that the ROG framework (Schoenfeld, 2010) is instrumental in identifying and understanding the factors that shape how prospective secondary mathematics teachers develop this network of relationships.

We utilized the ROG framework to describe and elucidate how groups of prospective teachers adapted tasks in accordance with inquiry-based principles. In addressing our research question regarding the degree of consistency in prospective teachers' decisions with respect to inquiry-based principles and student learning objectives, we analyzed the interplay among the resources, goals, and orientations manifested in their decision-making processes during task modification. Our findings affirm the utility of the ROG framework as a theoretical construct for investigating prospective teachers' learning of essential practices within mathematics teacher education programs. It facilitated the identification of critical features characterizing this learning process, particularly by highlighting the significance of teacher orientations in shaping decision-making.

The results indicate that when prospective teachers articulate students' specific mathematical practices as learning objectives, they are more successful in consistently employing inquiry-based teaching principles to modify tasks and inform their adaptations. Conversely, when they fail to prioritize these mathematical practices—such as generating specific cases, organizing information, systematically analyzing, recognizing relations and patterns, or formulating conjectures and generalizations—they encounter challenges in consistently utilizing inquiry-based teaching principles as resources. In such instances, the prospective teachers' decision-making during task modification revealed several logical contradictions between their intended learning goals and the decisions made based on selected inquiry-based principles. These challenges in aligning learning objectives with task modifications may be attributed to the prospective teachers' orientations.

The study's findings suggest that, in certain cases, prospective teachers' decisions regarding goal-setting are influenced more by their orientations than by the inquiry-based learning principles at their disposal. The identified characteristics of prospective teachers' learning indicate that task modification is shaped not only by the resources available to them but also by their orientations concerning mathematics teaching and learning. These orientations appear to influence how they establish students' learning goals (Ayalon et al., 2021; Dietiker et al., 2018; Lee et al., 2019) and facilitate prospective teachers' learning of core practices (Jacobs & Spangler, 2017). Thus, the ROG framework (Schoenfeld, 2010) allowed us to consider the interaction between inquiry-based teaching principles as resources and prospective teachers' orientations towards mathematics learning as a crucial explanatory factor in task modification learning.

We will now shift our focus to mathematical task modification as a fundamental practice that prospective teachers must learn. The findings of this study elucidate the characteristics of how prospective teachers utilize inquiry-based teaching principles as tools for task modification. Moreover, we propose that prospective teachers' learning of task modification aimed at enhancing student engagement with specific mathematical practices is intrinsically linked to the application of a theoretical framework as a tool (Ivars et al., 2020; Lee et al., 2023; Parrish et al., 2022). Modifying tasks to promote sense-making and active student engagement—through counterexamples or conjecturing patterns in a dataset—constitutes goal-oriented behavior supported by specific knowledge. When prospective teachers fail to identify the specific mathematical practices that could be cultivated, they struggle to coherently reflect the principles of inquiry-based mathematics teaching.

The way prospective teachers analyze and modify tasks to improve students' learning opportunities is influenced by various factors: their attention to task characteristics (both limitations and possibilities), their interpretations of these characteristics in light of inquiry-based principles, and how these interpretations affect their modification decisions. When prospective teachers neglect to address and interpret task limitations, they experience difficulties in coherently modifying the tasks based on inquiry-based principles. We argue that the analysis and modification of mathematical tasks as a core practice is initially contingent on how prospective teachers engage with task limitations, which may also be explained by their orientations and beliefs. This aspect serves as evidence of prospective teachers' comprehensive understanding of the relationships between inquiry-based principles and students' learning goals. However, additional research is warranted across various task types and contexts—such as field experiences in teacher preparation programs—to ascertain the developmental levels of task modification practices. Furthermore, more investigation is needed on how the ROG framework can enhance our understanding of prospective teachers' interactions with teaching resources, particularly when they collaboratively respond to tasks, as demonstrated in this study. We posit that the ROG framework (Schoenfeld, 2010), applied to the collaborative efforts of prospective teachers, can augment our approach to understanding their interactions with curricular materials (Dietiker et al., 2018) and be complemented by other theoretical perspectives that examine teachers' collaborative engagement with curricular resources (Trouche, Gitirana et al., 2019; Trouche, Gueudet et al., 2019).

Prospective teachers' orientations concerning the nature of school mathematics, and the processes of mathematical learning may, in some instances, serve as barriers to the acquisition of a core practice wherein resources, goals, and orientations are interwoven. Thus, further exploration is necessary to facilitate prospective teachers in aligning their beliefs with teaching innovations during collaborative work.

Finally, given that prospective teachers engaged collaboratively on only two tasks, our findings are limited to the specific context of this study. Therefore, further research is essential to augment our findings and understand the alignment between intended learning goals during task modifications and prospective teachers' orientations as they learn to apply theoretical knowledge. This understanding would aid teacher educators in better comprehending the conditions under which prospective teachers utilize knowledge as resources—such as inquiry-based principles or the cognitive demands of tasks—when developing core practices like task modification (Lee et al., 2023; Parrish et al., 2022). In conclusion, while we have provided a snapshot of how prospective secondary mathematics teachers make decisions regarding a professional task, it is equally important to examine how their decision-making evolves as they engage in this core practice during field interventions, such as university practicums or placements.

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Declarations

Author Contribution	:	MM: Conceptualization, Formal analysis, Investigation, Methodology, Writing - original draft, Writing - review & editing, Supervision.
		SL: Conceptualization, Formal analysis, Investigation, Methodology, Funding acquisition, Writing - original draft, Supervision.
		PS: Conceptualization, Formal analysis, Investigation, Methodology, Writing - original draft.

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