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Didactic strategy of linear algebra with collaborative learning in mathematics pedagogical training

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Abstract

This article details the design and implementation of a didactic intervention strategy aimed at enhancing the pedagogical skills of mathematics education students in the teaching of linear algebra. Given the challenges students often face in grasping abstract mathematical concepts, the intervention leverages the Anthropological Theory of Didactics and Collaborative Learning as its theoretical framework. The phases of the intervention's execution in the classroom, along with the key elements for successful implementation, are thoroughly outlined. The results indicate significant improvements in academic performance in the units' covering determinants, matrices, and systems of linear equations, while challenges persist in the unit on vector spaces. In terms of task genres, students showed proficiency in defining and calculating, although there was a lower level of mastery in demonstrating. The study concludes that, despite a general unfamiliarity with autonomous learning, the future educators were able to become active and effective contributors to their own mathematical knowledge construction through collaborative group work.

Keywords: Anthropological Theory of Didactics, Collaborative Learning, Didactics, Linear Algebra, Pedagogical Training

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The challenges associated with teaching and learning Linear Algebra have been a subject of research for several decades. Notably, in the 1990s, the Linear Algebra Curriculum Study Group (LACSG) in the United States served as a significant benchmark in this area. Concurrently, in Canada and Europe, researchers such as Dorier and Sierpinska led efforts to emphasize the universal recognition of the difficulties inherent in teaching Linear Algebra (Dorier, 2002; 2016; Dorier & Sierpinska, 2001). Both groups concur that one of the primary challenges in teaching and learning Linear Algebra is the formal, traditional approach typically used in classrooms (García et al., 2019).

Linear Algebra is a foundational course in many academic disciplines, including science, technology, and economics, due to its extensive applications (Strang, 2016; Abdurrahman et al., 2020). However, it is also perceived as one of the most difficult subjects at the college level (Rensaa et al., 2020; Dogan, 2018). According to Stuhlmann (2019), students often struggle with the abstract nature of algebraic structures and axiomatic methods, which are central to Linear Algebra. Therefore, Linear

Algebra, a fundamental course across various disciplines, is widely regarded as challenging for college students due to its abstract concepts and axiomatic methods.

Several researchers have highlighted significant challenges in learning Linear Algebra, particularly in students' struggles with verifying the correct application of symbols, selecting appropriate solution methods, and understanding core concepts, all exacerbated by the symbolic and algebraic language, with instructors facing the challenge of unifying these concepts across different contexts. Ferryansyah et al. (2018) analyzed the difficulties students face in learning Linear Algebra within a Mathematics Education program. They found that 91% of the students struggle with verifying whether the symbols, notations, or mathematical ideas they use are applied correctly. Similarly, Arnawa et al. (2019) report that students often encounter difficulties in selecting appropriate methods for solving systems of linear equations. Additionally, Berman and Shvartsman (2016), as well as Bagley and Rabin (2016), have identified a general weakness in students' understanding of core concepts in Linear Algebra and their related characteristics. These difficulties extend to interpreting problem statements and exercises, which are compounded by the symbolic and algebraic language (Osorio et al., 2023; León et al., 2019). The challenge for instructors is to unify concepts that students often perceive differently depending on the context—whether physical, geometric, numerical, graphical, or analytical (Pascual, 2020).

In Chile, Linear Algebra is a fundamental component of the national higher education curriculum, particularly within competency-based learning frameworks. Despite the assumed benefits of this curriculum, modeled after systems in more developed countries, student performance in Linear Algebra courses continues to reflect both cognitive and conceptual challenges (Cárcamo et al., 2017). These challenges have led to a search for alternative teaching methods that might better support student learning in this area.

González et al. (2019) underscore the urgent need to move beyond the didactic model that currently dominates mathematics education, which emphasizes passive learning where students are mere recipients of knowledge. Instead, there is a call for implementing changes that encourage both teachers and students to adopt new roles, promoting collaboration and the sharing of knowledge as the foundation of the educational process. Extensive research within Mathematics Education supports the use of collaborative approaches for learning various mathematical concepts (Nungu et al., 2023; Angulo-Vilca et al., 2021; Farfán-Pimentel et al., 2022; Orellana et al., 2020). These studies generally report some challenges but also very positive outcomes when collaborative work is integrated into teaching. However, despite the proven benefits of collaborative learning in other areas of mathematics, there is a notable gap in the literature regarding its application to the teaching of Linear Algebra at the undergraduate level.

This study aims to address this gap by exploring the teaching of Linear Algebra through a collaborative approach to foster meaningful learning and effective didactic transposition of knowledge. It focuses on the experiences of mathematics education students at a university in Chile. The research is guided by the following questions: What types of student tasks and outcomes are consolidated in the learning of Linear Algebra through a collaborative approach? and What transversal competencies do mathematics education students demonstrate when implementing a didactic strategy?

To answer these questions, a didactic experience was designed for a Linear Algebra course, conducted over a school semester from March to August. The primary objective was to design and implement a didactic intervention strategy for teaching Linear Algebra to mathematics education students. The specific objectives included describing the didactic strategy incorporating a collaborative methodology and analyzing mathematical content through a didactic lens that considers various tasks and techniques.

The theoretical framework for this research is built on two pillars: Chevallard's Anthropological Theory of Didactics (ATD) (Chevallard, 1999), which emphasizes the context in which mathematical activities occur and the institutions, as described by Gascón and Nicolas (2019), where these activities take place; and the Collaborative Learning Theory, representing the most prominent expression of educational socio-constructivism (Dillenbourg et al., 1996; Driscol & Vergara, 1997). These frameworks provide a robust foundation for understanding and evaluating the collaborative teaching strategies explored in this study.

Anthropological Theory of Didactics

The fundamental principle of the Anthropological Theory of Didactics (ATD) posits that all routinely performed human activities can be conceptualized through a unified model known as praxeology (Chevallard, 1999). Praxeology, also referred to as mathematical organization, serves as the primary framework for modeling mathematical activities and is comprised of two distinct levels:

1. The level of praxis or know-how, which encompasses a certain type of tasks and questions that are studied, as well as the techniques to solve them.
2. The level of logos or knowledge, in which the discourses that describe, explain and justify the techniques used are located, which are called technology. Within this domain of knowledge, a further level of description, explanation, and justification—referred to as the technology of technology—is recognized as theory.

As a methodological requirement, it is essential that the didactic experience addresses all stages of didactic transposition (Chevallard, 1985). This necessitates the incorporation of empirical data derived from each institution, which are composed of fundamental units of didactic analysis known as Mathematical Organization (MO). MO revolves around one or more types of mathematical tasks that lead to the development of mathematical techniques, which are then justified by mathematical technologies formulated within the framework of a mathematical theory. For further clarification, the didactic transposition framework (Chevallard, 1985) is illustrated in Figure 1.

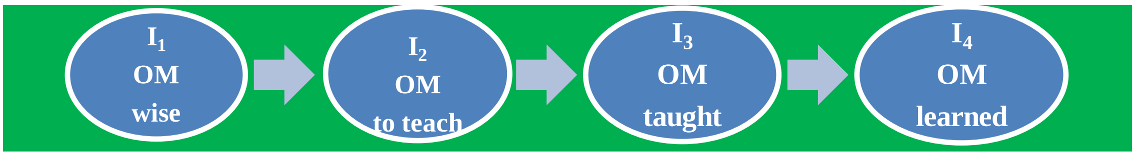

Figure 1. Scheme of Didactic Transposition

In Figure 1, I_1 represents the institution responsible for producing mathematical knowledge, I_2 denotes the noosphere, I_3 refers to the school institution, and I_4 signifies the study community, which is central to the didactic process. The acquired knowledge comprises praxeological elements that, by the end of the didactic process, will be integrated into the mathematical environment of the group. This knowledge can then be utilized by the study community to undertake new types of tasks and address emerging issues with relative ease. According to the methodology derived from the theoretical framework, our research involved reconstructing the Mathematical Organization to be taught and effectively imparted. The concepts of task type, technique, technology, and theory are fundamental for constructing any praxeology. Additionally, Chevallard (1999) introduces the notion of task genre, which refers to specific content characterized by different types of tasks. Task genres include categories such as defining, calculating, demonstrating, graphing, constructing, and others.

Collaborative Learning

In both academic and professional contexts, individuals are increasingly required to address problems not only independently but also collaboratively. According to the OECD (2017), collaborative problem solving offers several advantages, including the division of labor, diverse knowledge, varied perspectives and experiences, and mutual motivation, all contributing to more effective problem resolution and academic performance. Despite these benefits, only 8% of students from the countries evaluated by the OECD demonstrate high proficiency in this skill. These students are adept at managing group dynamics, ensuring team members adhere to their roles, resolving conflicts, identifying effective strategies, and monitoring progress towards solutions (OECD, 2017). Collaborative learning is recognized as a highly relevant and expanding field (Avci, 2022; Morge et al., 2020; Calder et al., 2021) and is integral to the competency-based approach. According to Asterhan and Schwarz (2016) and Gillies and Khan (2008), collaborative learning involves instructional arrangements where two or more students work together towards a common learning goal.

Numerous studies highlight the positive impact of collaborative learning on various aspects of education, including cognitive, metacognitive, affective-motivational, and social domains (Ahmad & Dogar, 2023; Er et al., 2020; Meijer et al., 2020; Sutama et al., 2021; Flores et al., 2015; Johnson & Johnson, 2009). Research by Chen and Preston (2022), Kyndt et al. (2014), Rohrbeck et al. (2003), and Roseth et al. (2008) indicates that students working in small groups tend to achieve better learning outcomes compared to those working individually. However, these benefits are contingent upon teachers making informed educational decisions (Kaendler et al., 2014; Van de Pol et al., 2010). Specifically, as students engage in collaborative work, it is essential for teachers to monitor the challenges students face and provide interventions when necessary (Van de Pol et al., 2015). Consequently, the teacher's role is pivotal in promoting effective student interaction (Abd, 2021; Klang et al., 2021).

For the purposes of this research, the five elements that characterize collaborative learning, as outlined by Driscoll and Vergara (1997), are considered. These elements are defined as follows:

1. Individual responsibility: all members are responsible for their individual performance within the group.
2. Positive interdependence: group members must depend on each other to achieve the common goal.
3. Collaboration skills: The skills necessary for the group to function effectively, such as teamwork, leadership, and conflict resolution.
4. Promoter interaction: group members interact to develop interpersonal relationships and establish effective learning strategies.
5. Group process: the group periodically reflects and evaluates its operation, making the necessary changes to increase its effectiveness.

METHODS

Study Type and Design

An exploratory-descriptive study was conducted (Hernández et al., 2014). The unit of analysis comprised

25 students, aged 18 to 20 years, who were enrolled in the compulsory Linear Algebra course during their second year in the Mathematics Pedagogy program at a Chilean university.

The Linear Algebra course spanned five months and included theoretical and practical classes held three times a week, with each session lasting 90 minutes. The course was led by a professor-researcher, who was responsible for directing the sessions, selecting the techniques to be employed, and addressing the various class activities. Both the professor and a teaching assistant were available to assist with student inquiries, both individually and in groups. The course covered five thematic units, each incorporating both theoretical and practical materials: Unit 1: Matrices and Systems of Linear Equations; Unit 2: Determinants; Unit 3: Vectors in the Plane and in Space; Unit 4: Vector Spaces; and Unit 5: Linear Transformations.

Development of the Didactic Strategy

Firstly, the theoretical material was designed by the teacher-researcher and included theorems, definitions, and fundamental propositions necessary for the study of Mathematical Organization. The class materials were structured with an introduction outlining the key theorems, definitions, and propositions required for the various types of tasks, accompanied by exercises and solved problems. Additionally, the material was developed alongside the planned activities and assessment methods, which were communicated to the students to clarify the purpose and procedures of the methodology. Consequently, what was traditionally a lecture-based class evolved into an open forum for dialogue, fostering interactions both among students and between students and the teacher.

In the classroom, activities were designed to encourage students to explain their learning to one another. Each student was assigned a specific role within the group, which facilitated learning from diverse perspectives, providing and receiving peer support, and collaboratively deepening their understanding of the concepts in each unit. A collaborative work mode was implemented to enhance mathematical learning and foster a culture of collaborative learning. This approach aimed to actively involve students in the construction and development of their mathematical knowledge through three phases, which were carried out throughout the semester.

Phase 1

The first unit was delivered entirely by the teacher-researcher, marking the initial exposure to a specific type of task. During this phase, the classes focused on tasks encompassing the genres of defining, calculating, demonstrating, and graphically representing. The explicit mathematical competencies outlined in the syllabus and included in the student assessments involved developing task genres and integrating matrix algebra. Students were expected to identify various types of matrices and their properties, perform matrix operations, relate systems of linear equations to matrices, and classify and solve systems of linear equations using matrix techniques.

In teaching this unit and emphasizing the importance of individual responsibility (Driscoll & Vergara, 1997), each student received a Linear Algebra manual prepared by the teacher-researcher. This manual included the unit's content, beginning with an introduction to the subject, examples of exercises and real-life problems, and work guides. All of this material was included in the students' portfolios of evidence. At the end of the unit, students were encouraged to further their individual knowledge and foster their autonomy by accessing educational tutorial channels on YouTube, focused on mathematics and, specifically, Linear Algebra. Additionally, activities were designed to address certain tasks and techniques collaboratively, incorporating the technological elements discussed in class (technological block).

Phase 2

This phase encompasses several key elements: the exploration of different types of tasks, the development of a technological-theoretical environment that explains and justifies the techniques used, the application of these techniques which leads to the evolution of existing methods and the creation of new ones, and the institutionalization phase, which delineates and specifies the components of the constructed Mathematical Organization. According to the curriculum, the transversal competencies to be achieved by students include ethics, teamwork/communication, and interest/vocation, in alignment with the guiding norms for the Pedagogy in Secondary Education program in Chile: Pedagogical and Disciplinary Norms (MINEDUC, 2012).

These norms underscore the necessity for students to assume responsibility for their learning through both individual and collaborative efforts, utilizing cognitive, metacognitive, and socio-affective strategies to advance their academic and professional growth. To align with the goal of preparing future secondary school mathematics teachers, students were grouped to undertake mathematical tasks aimed at providing practical experience with mathematical processes. The instructor designated four units—namely, determinants, vectors in the plane and space, vector spaces, and linear transformations—across four formally organized groups, which were voluntarily assembled by the students. This strategy was designed to enhance the group process dimension (Driscoll & Vergara, 1997) and to foster both oral and written mathematical communication through active discussion and idea exchange among students. This preparatory stage was essential before each group presented their findings on the mathematical subunits, which in turn impacted the group evaluation.

Each group member was expected to contribute to the collective objective, thereby reinforcing the dimension of positive interdependence (Driscoll & Vergara, 1997). This collaborative effort ensured that each participant played a role in achieving the group's common goal, which was crucial for effective group performance and evaluation.

The study program explicitly outlined and assessed mathematical competencies including the definition and properties of determinants, as well as their application in calculating inverse matrices and solving systems of linear equations with a matching number of unknowns and equations. Students were also required to identify and understand vectors in both planar and spatial contexts, encompassing their algebraic structure, properties, and representations. Furthermore, they needed to recognize vector spaces, including their properties and components, identify linear transformations between vector spaces along with their attributes, and compute eigenvectors and eigenvalues associated with these transformations. This comprehensive approach was designed to promote both individual and collective responsibility for assigned tasks, enhance mathematical communication through both oral and written means, and support the delivery of individual presentations on each mathematical subunit.

Additionally, students were encouraged to incorporate technological resources into their presentations to augment their effectiveness. An example of this is depicted in Figure 2, which demonstrates a technologically supported presentation on vector spaces. This emphasis on technology aimed to facilitate a deeper understanding of the material and improve overall presentation quality.

Figure 2. Oral Exhibition with Technological Support

In addition to delivering oral presentations, each group was tasked with conducting workshops to further explore the study unit for which they were assigned. These workshops were selected freely and by consensus within each group, with the aim of reinforcing the knowledge presented by the other groups, as illustrated in Figure 3. Based on the evaluation of each group's performance and the effectiveness of the resources utilized—within the framework of the promoter interaction dimension (Driscoll & Vergara, 1997)—groups had the option to revise their technological aids for subsequent oral presentations.

Figure 3. Workshops Elaborated by Group

At the end of the learning unit, each group was required to develop a summative assessment on the material they had taught. This assessment was administered during a 90-minute class session and had been reviewed and approved by the teacher-researcher in advance. The knowledge that their peers would be evaluating their work significantly heightened the students' engagement during the presentations of the content and increased the frequency of consultations with the presenting groups. The rigor of these assessments was high, which necessitated a more consistent study effort from the rest of the students.

Throughout this process, students received continuous, active support to discuss, articulate, and document their understanding of mathematical concepts. The various activities were designed to motivate students and empower them to take ownership of their learning. This approach facilitated interpersonal interactions and led to the development of effective learning strategies for Linear Algebra.

Initially, students exhibited discomfort with the activities due to their departure from conventional learning experiences, despite the curriculum being designed to be competency-based. However, following several individual presentations as part of their group assignments, students gradually acclimated to the new format. They began to exhibit enthusiasm for collaborative work, actively sharing ideas and engaging with their peers. The newly established environment thus fostered enhanced collaboration, critical thinking, and communication skills, contributing to a more dynamic classroom atmosphere.

During the presentations, it became common for students to seek further clarification on concepts from their peers, with responses varying in clarity. In such instances, both the teacher-researcher and the teaching assistant played active roles in supporting and guiding the discussion.

Phase 3

This phase pertains to the assessment of the developed praxeology. At the conclusion of each of the five instructional units, the students were evaluated by the teacher-researcher. The units covered were Unit 1: Matrices and Systems of Linear Equations; Unit 2: Determinants; Unit 3: Vectors in the Plane and in Space; Unit 4: Vector Spaces; and Unit 5: Linear Transformations. Each test was thoroughly reviewed and graded, with the teaching assistant's involvement, to provide comprehensive feedback.

In addition to these summative written tests, the evaluation of student performance by the teacher-researcher encompassed a portfolio of evidence for each student, oral presentations, and peer evaluations of the four groups. This multi-faceted approach ensured a holistic assessment of the students' understanding and application of the material.

RESULTS AND DISCUSSION

The Mathematical Organization that was effectively taught emerged as a direct outcome of the study process and was the result of the instructional practices implemented by the teacher-researcher in the classroom. This organizational framework reflects the pedagogical strategies and methodologies employed throughout the study. Subsequently, the various types of tasks (T) and subtasks (t) associated with the four genres (G) integrated into the classes are defined and exemplified. These classifications illustrate the diverse forms of tasks and their respective roles within the instructional framework.

- G1 *Define*: It encompasses those types of tasks that aim to reconstruct certain mathematical definitions; the teaching activity focuses on such a reconstruction, and not only on reproducing the definitions. Example:

  T_1 Define a matrix of order (or size) m×n over IR

  $t_{1,1}$ Define a diagonal matrix

  $t_{1,2}$ Define a symmetric matrix

- G2 *Demonstrate*: Includes those types of tasks that require the formulation of a sequence of organized statements, according to certain rules. Example:
  T_2 Demonstrate properties that relate subsets of IR^n formed by linearly independent (l.i.) and linearly dependent (l.d.) vectors.
  $t_{2,1}$ Let be the set $\{x, y\}$ l.i., with x, y ∈ IR^n. Demonstrate that the sets: $\{x, ax + y\}$, $\{x - y, x + y\}$, are linearly independent.
  $t_{2,2}$ Let be the set $\{x, y\}$ l.i., with x, y ∈ IR^n. Demonstrate that the set $\{x - y, y - x\}$ is linearly dependent.
  $t_{2,3}$ Demonstrate if the following set is l.i. $\{(1,2,1), (2,1,-1), (-1,1,1)\}$

- G3 *Calculate*: It refers to tasks that imply carrying out certain procedures based on rules that are taken as true to obtain a result and predict some events within mathematics or other disciplines. Example:
  T_3 Calculate a vector equation for parallel and perpendicular lines.
  $t_{3,1}$ Calculate the vector equation of the line parallel to $2 = \frac{x-2}{2}, y = \frac{z+3}{-3}$ and containing the point $(2,3,-5)$.
  $t_{3,2}$ Calculate the vector equation of the line that passes through the origin and is perpendicular to the lines $(x, y, z) = (2 - 3t, -3, 4 + t)$ y $(x, y, z) = (t, 0, 3t)$
  $t_{3,3}$ Calculate the vector equation of the line that cuts perpendicularly to the line recta $\frac{x+1}{3} = \frac{3-y}{2}$, $z = 1$ at $(-1,3,1)$.

- G4 *Representing graphically*: Includes tasks that involve drawing diagrams and graphic representations. Example:
  T_4 Draw the graph and mark the region that corresponds to the vectors that satisfy the given condition.
  $t_{4,1}$ To graph the points (a, b)such that $(a, b) = t(-2,3)$ where $1 \leq t < 3$, $t \in IR$
  $t_{4,2}$ To graph the points (x, y) such that $(x, y) = t(-2,3) + s(1,4)$ where $0 \leq t \leq 1$ and $0 \leq s < 1$.

The following section provides a detailed account of the primary results obtained from the final assessments across the five Linear Algebra units. These results include the types of tasks (T) and subtasks (t) related to the four genres (G)—define, demonstrate, calculate, and graph—that were integral to the lessons, as illustrated in Figure 4.

In the unit on determinants, a high percentage of students demonstrated proficiency in solving the tasks, resulting in the highest performance rate of 83.8% among the five units studied. This was followed by the unit on matrices and systems of linear equations, with a performance rate of 76.4%. In the latter unit, students encountered challenges such as the application of invalid properties, which appeared to be adaptations of valid properties from other contexts, and the use of valid but irrelevant arguments. Some students also faced minor difficulties with matrix operations.

In the unit on vectors in the plane and in space, which achieved a performance rate of 74%, students encountered fewer difficulties in understanding certain concepts due to their prior knowledge of analytical geometry. Conversely, the unit on vector spaces recorded the lowest performance rate of 50.1%. This lower performance was attributed to some students' inability to effectively solve tasks or their use of inappropriate techniques. Despite engaging with the concept of linear combinations and building upon their understanding of vectors as geometric entities to develop the formal definition of linear independence, students struggled with various aspects of the unit.

Tasks related to the concept of linear independence involved subsets of finite-dimensional vector spaces, such as n-tuples of real numbers, polynomials with real coefficients of degree less than or equal to n, and real matrices of order m×n, along with their standard operations. Students generally succeeded in determining whether a subset was linearly dependent or independent. However, many failed to recognize that the zero vector in the homogeneous equation used to analyze linear independence is the null vector of an abstract vector space. This challenge became apparent when students either confused the zero vector with the zero scalar or misinterpreted it as the null vector of n components. Additionally, students struggled with appropriately formalizing and justifying their answers, despite having a grasp of the concept of linear independence.

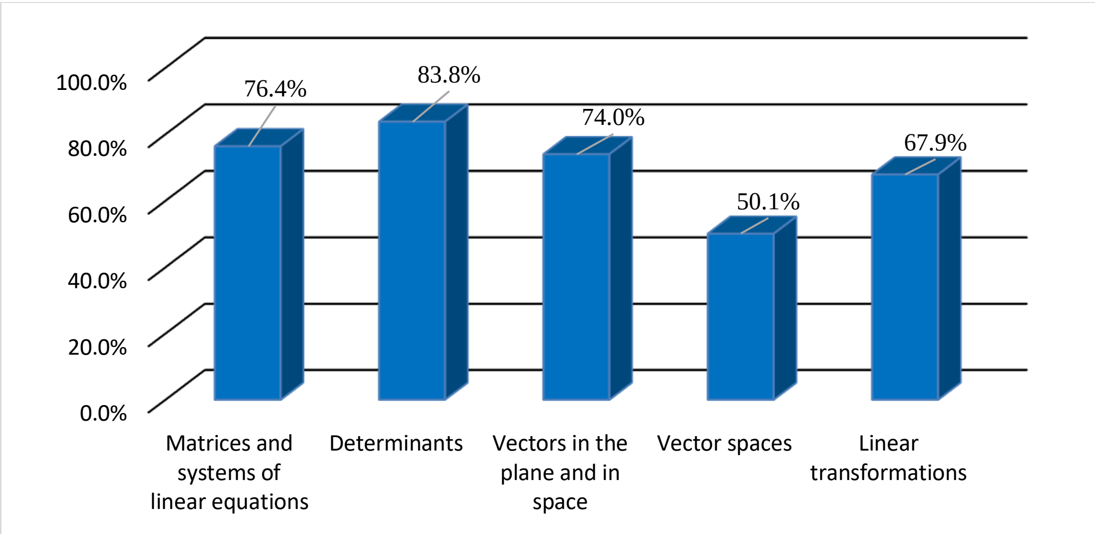

Figure 4. Student Performance in Study Units

The misrepresentation errors related to the elements of a vector subspace suggest that students may struggle with the abstraction required to understand that a vector is not limited to being an element of $\mathbb{R}^n$ but can also represent a polynomial, a matrix, or other elements within a given vector space. This difficulty in grasping the broader concept of vectors likely led students to simplify the abstraction during the evaluation, defaulting to familiar elements of $\mathbb{R}^n$ and, consequently, making incorrect representations.

The unit on linear transformations recorded the second lowest performance rate, with 67.9% of students achieving success. A common error observed was the students' tendency to place values in a proportional relationship using linear combinations, indicating a preference for procedural approaches over structural considerations of the model's properties. This pragmatic thinking led them to assume a proportional model without adequately establishing the conditions for its application within the real model. Additionally, difficulties were noted concerning the linear properties inherent to the rotation of a plane, where students often relied on a flawed, physically-based representation of "rotation." They tended to conceptualize plane rotation within the context of Cartesian coordinates, utilizing systems of linear equations. Because expressing rotation through linear combinations more effectively captures its linear nature, which remains invariant under a change of basis, students likely associated this concept with a change in linear coordinates within the affine plane.

In the tasks related to coordinate changes in the affine plane $\mathbb{R}^2$, students found it relatively straightforward to apply trigonometric identities to derive the algebraic expression for the coordinates of the point $Q = R_\theta(P)$. This ease is largely due to their familiarity with the foundational concepts of analytical geometry and trigonometry, which were part of their prior learning.

The average results for these tasks, categorized according to the four genres, are presented below. These genres group the types of tasks and subtasks addressed throughout the instructional experience and were incorporated into the evaluations of the five units, as illustrated in Figure 5.

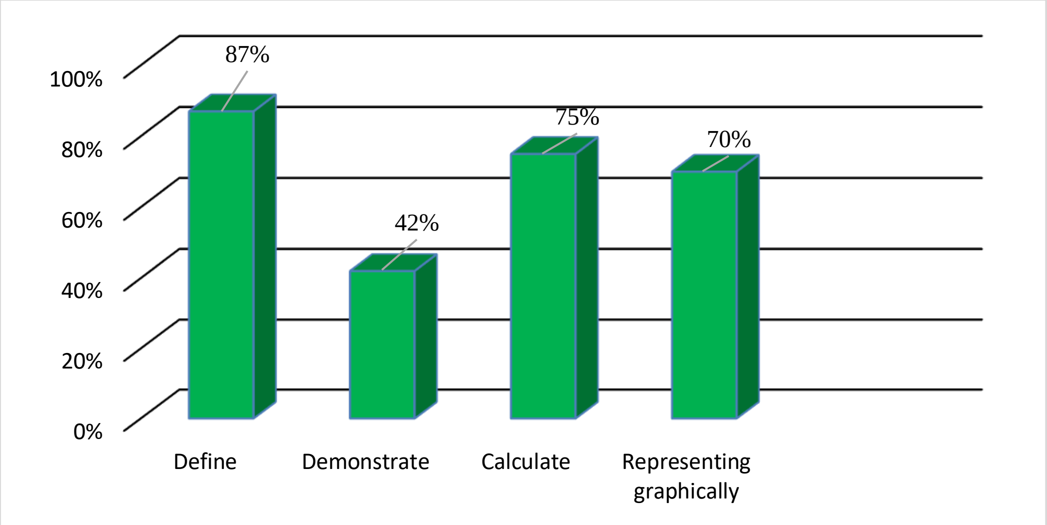

Figure 5. Results of the Genres that Group Types of Tasks

In general, high average scores were observed in the "define," "calculate," and "represent graphically" genres, while significant challenges were noted in the "demonstrate" genre. Specifically, in the final test for the vector spaces unit, the analysis of tasks within the "calculate" genre revealed that a large percentage of students (89%) either did not attempt the task or employed inappropriate techniques, with only 6% solving the task correctly. This outcome contrasts sharply with the results from the unit on determinants, where 90% of students successfully solved the tasks. The disparity can be attributed to difficulties in both the algebraic manipulation of expressions and the conceptual understanding required in the vector spaces unit.

In the "demonstrate" genre, as depicted in Figure 5, students' performance averaged 42% for correctly completed tasks. In the vector spaces unit's final test, a substantial 72% of students either failed to attempt the proof or completed it incorrectly. Only 8% successfully carried out the proof, while 10% managed to prove the proposition correctly in one direction, indicating some level of understanding. This challenge may stem from the fact that demonstration is a procedural aspect of mathematics that is gradually developed over time throughout various stages of schooling.

Discussion

The challenges associated with learning Linear Algebra compel educators to reconsider the ineffective approaches often employed in mathematics classrooms, which have historically yielded disappointing outcomes (Osorio et al., 2023; Arnawa et al., 2019; Rensaa et al., 2020; Abdurrahman et al., 2020; García et al., 2019). To address these issues, it is essential to develop and refine didactic strategies that can enhance teaching effectiveness. One promising approach is the incorporation of collaborative learning as a methodological strategy.

This collaborative approach is particularly relevant within mathematics courses, where the necessary solutions and techniques for defining Mathematical Organizations can be found. The collaborative didactic experience conducted with future mathematics teachers at a Chilean university serves as a strong example of this strategy in action, demonstrating its potential to improve the teaching and learning of Linear Algebra.

The results showed that the performance percentages across the five units of Linear Algebra ranged from 50.1% to 83.8%, with the lowest performance observed in the vector spaces unit and the highest in the determinants unit. When examining the average results based on the four genres, which categorize the types of tasks present in the classes, high average scores were achieved in the "define," "calculate," and "represent graphically" genres, while the "demonstrate" genre saw lower average achievement. Overall, the majority of students were empowered both individually and as a group, gaining substantial knowledge of linear algebra through the incorporation of didactic transposition stages and an effective collaborative work methodology.

These findings align with research from various regions that highlight the benefits of collaborative learning. For instance, Chen and Preston (2022) found that collaborative learning enhanced the critical thinking skills of second-year students at a Chinese university, although they suggested that instructors should employ specific techniques to further develop team members' collaborative skills. Similarly, Fauzi et al. (2021) in Indonesia determined that collaborative learning techniques were effective in improving critical thinking skills among high school students engaged in group research.

Pozzi et al. (2023) conducted a study in Italy aimed at identifying areas where current collaborative teaching strategies could be enhanced to increase their effectiveness and broader adoption in schools. The study revealed that Italian teachers do incorporate collaborative learning activities in various formats—face-to-face, blended, and fully online—with statistically significant differences in the approaches and technological tools utilized. However, the findings also highlighted that teachers' design decisions often diverge from the best practices recommended by the collaborative learning research community, indicating a need for better alignment between theory and practice.

In relation to the challenges encountered in the vector spaces unit, it is important to acknowledge that the concept of a vector space represents an abstraction of already abstract mathematical objects, characterized by shared properties such as vectors, n-tuples, polynomials, series, or functions. The literature consistently identifies obstacles to understanding this concept. Dorier and Sierpinska (2001) note that the abstract nature of vector spaces makes them a central yet challenging concept in Linear Algebra. Mutambara and Bansilal (2018) describe this difficulty as a formalism obstacle, while Martín et al. (2014) and Oktaç and Gaisman (2010) attribute students' struggles to the abstract nature of the concept. Parraguez and Oktaç (2010) further emphasize the need to formalize prior concepts—such as sets, functions, and binary operations—as a prerequisite for successfully learning about vector spaces.

In analyzing the theoretical constructs essential for collaborative work, as outlined by Driscoll and Vergara (1997), these constructs were closely integrated with the tasks performed by students in the Linear Algebra units. Regarding positive interdependence, the key insight was that the presence of a single indicator was insufficient for establishing genuine collaboration. Instead, multiple indicators of positive interdependence needed to be expressed. For instance, effective collaboration was evidenced not just by sharing resources but also by assuming complementary roles. Merely sharing resources did not suffice to create truly collaborative work. The success of all students depended on the collective effort, aligning with Zariquiey's (2019) assertion that positive interdependence leaves no room for individualism.

In this context, the study by Aristizabal-Almanza et al. (2018) further supports the importance of positive interdependence in collaborative learning. Their findings indicate that team members respect and value each other's individual contributions, which in turn enhances their communication and listening skills. This suggests that fostering positive interdependence within groups is crucial for improving the overall effectiveness of collaborative learning in mathematics.

Detecting individual responsibility within group discussions and presentations was somewhat challenging, although all members responded to their assigned tasks to some extent. Basantes and Santisteban (2019) similarly argue that evaluations should not only assess the group as a whole but also consider the actions of each member based on their responsibilities and duties. Additional insights were gained from student portfolios and oral presentations from the four groups, which highlighted the role of individual responsibility in the overall work. These sources also underscored the importance of feedback mechanisms, such as peer evaluations and the construction of peer assessments, in raising awareness of individual responsibility.

In terms of promotive interaction, the most prominent indicator was the contribution to problem-solving and the interpretation of tasks. This finding aligns with Lazareva's (2015) study, which identified other factors that enhance student engagement in collaborative work, including the significance of the task and its impact on course grades. When a group member contributed meaningfully to problem resolution, other members typically responded with evaluations, opinions, or corrections, fostering a collaborative environment where ideas were actively exchanged and refined.

Another element observed in the messages exchanged during instances of promotive interaction was the presence of affective components. Students expressed a sense of confidence in delivering their contributions, knowing they had a reliable group to depend on and recognizing that their performance impacted their peers. This support system allowed students to voice their doubts and concerns and provided them with the necessary encouragement to collaborate towards a shared goal. Järvenoja et al. (2020) highlight that while collaborative learning approaches are generally beneficial, they can also be motivationally and emotionally demanding. Students are required to engage in higher-level thinking and interaction while managing increased responsibility for both individual and group learning processes (Mäkitalo-Siegl & Fisher, 2013).

Consequently, many of the challenges students face stem from cognitive obstacles with socio-emotional and motivational roots (Järvenoja et al., 2018). There is a recognized need to focus on supporting motivation and emotional well-being within groups while establishing a common ground for effective collaboration (Ludvigsen, 2016). This emphasis on socio-emotional support is crucial for fostering a productive and collaborative learning environment.

One frequently observed indicator of promotive interaction was the appreciation of another group's effort, which was commonly seen during both the initiation and ongoing stages of group interventions. This indicator was notably influenced by the performance of group leaders, who were responsible for ensuring active participation from all members. Effective collaboration skills were integral throughout the development of the activities related to the subject. Catalán et al. (2023) emphasize that collaboration skills are crucial and are acquired progressively through practice. Skills, such as raising one's hand, requesting the floor, respecting others' opinions, and encouraging the group are rooted in broader transversal competencies including empathy, trust, respect, self-awareness, and emotional regulation.

Finally, in evaluating the group's performance, students had the opportunity to review and reflect on their work through feedback from other groups, which facilitated necessary adjustments to enhance effectiveness. This aligns with Espinal et al. (2022), who suggest that such reflective practices contribute to interactions at various cognitive levels and stimulate the development of learning capacities. Stahl et al. (2014) and Pozzi et al. (2023) also emphasize that groups must work towards a common task during educational activities. A shared and concrete goal fosters group discussions, and the need to achieve consensus promotes the exchange of ideas and the construction of new knowledge.

The analysis of research results, prior studies, and the theoretical framework underscores the importance of addressing the intrinsic difficulty of mathematical subjects, particularly for student teachers. This approach may facilitate their learning and adaptation to university life through a methodology with described characteristics, which could be applicable to other areas of mathematics or even different disciplines. The research offers a detailed examination of various genres, types, and techniques of assignments, contributing to a deeper understanding of teaching and learning processes. Additionally, it provides a practical example of implementing a collaborative learning strategy in a real classroom setting.

CONCLUSION

The objective of this study was to design and implement a didactic intervention strategy within the regular framework of a Linear Algebra course for mathematics pedagogy students, utilizing a collaborative learning methodology. Employing the Anthropological Theory of Didactics (ATD), which facilitated the characterization of Mathematical Organizations, we identified and analyzed genres of tasks, types of tasks, and techniques based on the theoretical constructs and their integration (Chevallard, 1985). Our research concentrated on tasks evaluated during the study units and the students' responses to these tasks. This article provides a comprehensive account of the didactic experience, illustrating the application of collaborative learning in the context of teaching Linear Algebra within a competency-based curriculum.

The study outlines each phase of implementing the didactic intervention in the classroom, specifying the essential elements for effective execution. The research findings indicate that the application of this strategy resulted in enhanced learning outcomes and improved performance in evaluations throughout the course. The aim is to offer an alternative teaching approach at the higher education level that contrasts with traditional methods. By integrating the Anthropological Theory of the Didactic (ATD) and collaborative learning, the approach facilitates a more active role for students, making subjects like Linear Algebra more accessible and less challenging.

Declarations

Author Contribution : VD: Conceptualization, Writing - Original Draft, Methodology, Investigation, Editing and Visualization, and Writing-Review & Editing.
AHD: Conceptualization, Formal analysis, and Methodology.
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REFERENCES

Abd, Y. M. (2021). The effect of the collaborative learning technique on students 'educational performance in math. *Journal for the Mathematics Education and Teaching Practices*, *2*(2), 93-103. http://openaccess.izmirakademi.org/paper/1815/66397/1052185

Abdurrahman, M.S., Umaru, W., Abdullah, A.H., & Osman, S. (2020). Effect of peer tutoring strategy on students' academic performance in a polytechnic linear algebra classroom. *Journal of Advanced Research in Dynamical and Control Systems*, *12*(3), 415-422. https://doi.org/10.5373/JARDCS/V12I3/20201209

Ahmad, M., & Dogar, A. H. (2023). Effect of collaborative learning on conceptual understanding ability in mathematics among 5th grade neglected children. *Annals of Human and Social Sciences*, *4*(2), 205-213. https://doi.org/10.35484/ahss.2023(4-II)19

Angulo-Vilca, P. (2021). El aprendizaje colaborativo virtual para la enseñanza de la matemática. *Dominio de las Ciencias*, *7*(1), 253-267. http://orcid.org/0000-0003-4056-1892

Aristizabal-Almanza, J., Ramos-Monobe, A., & Chirino-Barceló, V. (2018). Aprendizaje activo para el desarrollo de la psicomotricidad y el trabajo en equipo. *Revista Electrónica Educare*, *22*(1), 1-26. https://doi.org/10.15359/ree.22- 1.16

Arnawa, Y., Yerizon., Y. & Nita, S. (2019). Errors and misconceptions in learning elementary linear algebra. *Journal of Physics: Conference Series*, *1321*(2), 022095. https://doi.org/10.1088/1742-6596/1321/2/022095

Asterhan, C. S. C., & Schwarz, B. B. (2016). Argumentation for learning: Well-trodden paths and unexplored territories. *Educational Psychologist,* *51*(2), 164-187. https://doi.org/10.1080/00461520.2016.1155458

Avci, B. (2022). Collaborative learning within critical mathematics education. In M. Godoy, & O. Skovsmose (Eds.), *Landscapes of investigation, contributions to critical mathematics education* (pp.115-132). Open Book Publishers. https://doi.org/10.11647/OBP.031

Bagley, S., & Rabin, J. M. (2016). Students' use of computational thinking linear algebra. *International Journal of Research in Undergraduate Mathematics Education*, *2*(1), 83-104. https://doi.org/10.1007/s40753-015-0022-x

Basantes, A., & Santiesteban, I. (2019). Aprendizaje cooperativo, estudio diagnóstico desde la perspectiva de los docentes. *Revista Conrado,* *15*(67), 200-204. http://scielo.sld.cu/scielo.php?script=sci_abstract&pid=S1990-86442019000200200

Berman, A., & Shvartsman, L. (2016). Definitions are important: The case of linear algebra. European *Journal of Science and Mathematics Education*, *4*(1), 26-32. https://doi.org/10.30935/scimath/9450

Calder, N., Jafri, M., & Guo, L. (2021). Mathematics education students' experiences during lockdown: Managing collaboration in eLearning. *Education Sciences*, *11*(4), 191. https://doi.org/10.3390/educsci11040191

Cárcamo, A., Fortuny, J., & Gómez, J. (2017). Mathematical modelling and the learning trajectory: tools to support the teaching of linear algebra. *International Journal of Mathematical Education in Science and Technology*, *48*(3), 38-352. https://doi.org/10.1080/0020739X.2016.1241436

Catalán, M. R., Figueroa, M. G., & Espinoza, R. M. (2023). Aprendizaje cooperativo, trascendiendo el aula convencional. *Horizontes. Revista de Investigación en Ciencias de la Educación*, *7*(27), 86–98. https://doi.org/10.33996/revistahorizontes.v7i27.499

Chen, Y., & Preston, J. C. (2022). Improving EFL students' critical thinking skills through organization development intervention strategies at Zhejiang Yuexiu University in China. *ABAC ODI Journal Vision*, *9*(1), 181-203. https://doi.org/10.14456/abacodijournal.2021.25

Chevallard, Y. (1985). *La transposition didactique*. La Pensée Sauvage. https://Desktop/11DID_Chevallard_Unidad_3.pdf

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*, *19*(2), 221–266. https://revue-rdm.com/1999/l-analyse-des-pratiques/

Dillenbourg, P., Baker, M., Blaye, A. & O'malley, C. (1996) The evolution of research on collaborative learning. In E. Spada, & P. Reiman (Eds.), *Learning in Humans and Machine: Towards an Interdisciplinary Learning Science* (pp. 189-211). Elsevier. https://www.researchgate.net/publication/32231168_The_evolution_of_research_on_collaborative_learning

Dogan, H. (2018). Mental schemes of: Linear Algebra visual constructs. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.), *Challenges and Strategies in Teaching Linear Algebra* (pp. 219-239). Springer. https://doi.org/10.1007/978-3-319-66811-6_10

Dorier, J.L. (2002). Epistemological analysis of the genesis of the theory of vector spaces. In J.L. Dorier (Ed.), *On the teaching of Linear Algebra* (pp.3-81). Springer. https://doi.org/10.1007/0-306-47224-4_1

Dorier, J.L. (2016). *Duality between formalism and meaning in the learning of linear algebra*. Universitäts bibliothek Kassel. https://archive-ouverte.unige.ch/unige:85576

Dorier, J.L., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level- An ICMI Study* (pp. 255-273). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47231-7_24

Driscoll, M., & Vergara, A. (1997). Nuevas tecnologías y su impacto en la educación del futuro. *Pensamiento Educativo*, *21*(2), 81-99. https://ojs.uc.cl/index.php/pel/article/view/24589

Er, E., Dimitriadis, Y., & Gasevic, D. (2020). A collaborative learning approach to dialogic peer feedback: A theoretical framework. *Assessment & Evaluation in Higher Education*, *46*(2), 1-15. https://doi.org/10.1080/02602938.2020.1786497

Espinal, C. A., Tapia, A., Guerra, D. L., & Martel, L. V. (2022). Aprendizaje colaborativo para la mejora del pensamiento crítico en estudiantes de secundaria. *Horizontes. Revista de Investigación en Ciencias de la Educación*, *6*(26), 1951–1960. https://doi.org/10.33996/revistahorizontes.v6i26.464

Farfán-Pimentel, J. F., Crispín Rommel, L., Carreal-Sosa, C. L., Quiñones-Castillo, K. G., & Farfán-Pimentel, D. E. (2022). Aprendizaje colaborativo en el desarrollo de competencias matemáticas en estudiantes de secundaria. *Ciencia Latina. Revista Científica Multidisciplinar*, *6*(5), 5335-5357. https://doi.org/10.37811/cl_rcm.v6i5.3505

Fauzi, F., Erna, M., & Linda, R. (2021). The effectiveness of collaborative learning through techniques on group investigation and think pair share students' critical thinking ability on chemical equilibrium material. *Journal of Educational Sciences*, *5*(1), 198-208. https://doi.org/10.31258/jes.5.1.p.198-208

Ferryansyah, E., Widyawati, E., & Rahayu, S.W. (2018). The analysis of students' difficulty in learning Linear Algebra. *Journal of Physics: Conference Series*, *1028*(1), 012152. https://doi.org/10.1088/1742-6596/1028/1/012152

Flores, M.A., Veiga, A.M., Barros, A., & Pereira, D. (2015). Perceptions of effectiveness, fairness and feedback of assessment methods: A study in higher education. *Studies in Higher Education*, *40*(9), 1523–1534. https://doi.org/10.1080/03075079.2014.881348

García-Hurtado, M. M. García-Pupo and R. Poveda-Chaves (2019). Linear algebra learning focused on plausible reasoning in engineering programs. *Visión Electrónica*, *13*(2), 322-330. https://doi.org/10.14483/22484728.15164

Gascón, J., & Nicolás, P. (2019). Research ends and teaching ends in the anthropological theory of the didactic. *For the Learning of Mathematics*, *39*(2), 42–47. https://www.jstor.org/stable/26757472

Gillies, R. M., & Khan, A. (2008). The effects of teacher discourse on students' discourse, problem-solving and reasoning during cooperative learning. *International Journal of Educational Research*, *47*(6), 323-340. https://doi.org/10.1016/j.ijer.2008.06.001

González, M, Vera, K. & Romero, Y. (2019). Estrategias colaborativas como herramienta para el aprendizaje del álgebra lineal. *REDHECS*, *27*(14), 169-184. https://ojs.urbe.edu/index.php/redhecs/article/view/1341/2201

Hernández, R., Fernández, C., & Baptista, M. (2014). *Metodología de la investigación*. McGraw-Hill Interamericana.https://www.academia.edu/32697156/Hern%C3%A1ndez_R_2014_Metodologia_de_la_Investigacion

Järvenoja, H., Järvelä, S., Törmänen, T., Näykki, P., Malmberg, J., Kurki, K., Mykkänen, A., & Isohätälä, J. (2018). Capturing motivation and emotion regulation during a learning process. *Frontline Learning Research*, *6*(3), 85-104. https://doi.org/10.14786/flr.v6i3.369

Järvenoja, H., Malmberg, J., Törmänen, T., Mänty, K., Haataja, E., Ahola, S., & Järvelä, S. (2020). A collaborative learning design for promoting and analyzing adaptive motivation and emotion regulation in the science classroom. *Frontiers in Education*, *5*, 111. https://doi.org/10.3389/feduc.2020.00111

Johnson, D. W., & Johnson, R. T. (2009). An educational psychology success story: Social interdependence theory and cooperative learning. *Educational Researcher*, *38*(5), 365-379. https://doi.org/10.3102/0013189X09339057

Kaendler, C., Wiedmann, M., Rummel, N., & Spada, H. (2014). Teacher competencies for the implementation of collaborative learning in the classroom: A framework and research review. *Educational Psychology Review*, *27*(3), 1-32. https://doi.org/10.1007/s10648-014-9288-9

Klang, N., Karlsson, N., Kilborn, W., Eriksson, P., & Karlberg, M. (2021). Mathematical problem-solving through cooperative learning—the importance of peer acceptance and friendships. *Frontiers in Education*, *6*, 710296. https://doi.org/10.3389/feduc.2021.710296

Kyndt, E., Raes, E., Lismont, B., Timmers, F., Dochy, F., & Cascallar, E. (2014). A meta-analysis of the effects of face-to-face cooperative learning. Do recent studies falsify or verify earlier findings? *Educational Research Review*, *10*, 133-149. https://doi.org/10.1016/j.edurev.2013.02.002

Lazareva, A. (2015). Promoting collaborative interactions in a learning management system. In M. Auer (Eds.), *Proceedings of the 2015 International Conference on Interactive Collaborative Learning* (pp. 421-430). https://doi.org/10.1109/ICL.2015.7318066

León, C., Coello, C., Goyes, K., Barzola, J., & León, L. (2019). Proceso enseñanza aprendizaje del algebra lineal. Sistematización e implicación en las carreras de la facultad de ciencias de la ingeniería de la Universidad Técnica Estatal de Quevedo. *Ciencias Sociales y Económicas*, *3*(2), 134–155. https://doi.org/10.18779/csye.v3i2.300

Ludvigsen, S. (2016). CSCL: Connecting the social, emotional and cognitive dimensions. *International Journal of Computer-Supported Collaborative Learning,* *11*(2), 115–121. https://doi.org/10.1007/s11412-016-9236-4

Mäkitalo-Siegl, K., & Fischer, F. (2013). Help seeking in computer-supported collaborative science learning environments. In S. A. Karabenick, & M. Puustinen (Eds.), *Advances in Help-seeking Research and Applications: The Role of Emerging Technologies* (pp.99-120). IAP Information Age Publishing. https://psycnet.apa.org/record/2013-32719-005

Martín, Á., Pérez, O., Blanco, R., & Casas, L. (2014). Los espacios vectoriales, como estructuras algebraicas, en el proceso de enseñanza aprendizaje del algebra lineal: Una propuesta de investigación. In P. Leston (Ed.), *Acta Latinoamericana de Matemática Educativa* (pp. 1073-1082). CLAME.

Meijer, H., Hoekstra, R., Brouwer, J., & Strijbos, J.W. (2020). Unfolding collaborative learning assessment literacy: A reflection on current assessment methods in higher education. *Assessment & Evaluation in Higher Education*, *45*(8), 1222-1240. https://doi.org/10.1080/02602938.2020.1729696

MINEDUC (2012). *Ministerio de Educación de Chile. Estándares orientadores para carreras de pedagogía en educación media*. LOM Ediciones Ltda. https://bibliotecadigital.mineduc.cl/handle/20.500.12365/2226

Morge, S. P., Schwartz, C. S., & Hargrove, T. (2020). Strategies and tools for promoting discourse during mathematics problem-solving in online settings. In P. Wachira & J. Keengwe (Eds.), *Handbook of Research on Online Pedagogical Models for Mathematics Teacher Education* (pp. 216–233). IGI Global. https://doi.org/10.4018/978-1-7998-1476-4.ch013

Mutambara, L. H. M., & Bansilal, S. (2018). Dealing with the abstraction of vector space concepts. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.), *Challenges and Strategies in Teaching Linear Algebra. ICME-13 Monographs* (pp.147–173). Springer. https://doi.org/10.1007/978-3-319-66811-6_7

Nungu, L., Mukama, E., & Nsabayezu, E. (2023). Online collaborative learning and cognitive presence in mathematics and science education. Case study of university of Rwanda, college of education. *Education and Information Technologies*, *28*, 10865-10884. https://doi.org/10.1007/s10639-023-11607-w

OECD (2017). *Collaborative problem solving. PISA in Focus*. OECD Publishing. https://doi.org/10.1787/cdae6d2e-en

Oktaç, A., & Gaisman, M. (2010). ¿Cómo se aprenden los conceptos de Algebra Lineal? *Revista Latinoamericana de Investigación en Matemática Educativa*, *13*(4), 373–385. https://relime.org/relime/index.php/numeros/todos-numeros/volumen-13/numero-especial-13-4-ii/499-201021d

Orellana, R., García, D., Mena, S., & Erazo, J.C. (2020). Aprendizaje colaborativo y matemáticas en Educación Básica Superior. *Episteme Koinonía*, *3*(1), 543-559. https://doi.org/10.35381/e.k.v3i1.1025

Osorio, V., Palomino, J., Huayhua, M., & Gambini, I. (2023). Enseñanza del algebra lineal en estudiantes universitarios. Horizontes. *Revista de Investigación en Ciencias de la Educación*, *7*(27), 380-387. https://doi.org/10.33996/revistahorizontes.v7i27.522

Parraguez, M., & Oktaç, A. (2010). Construction of the vector space concept from the view point of APOS theory. *Linear Algebra and its Applications*, *432*(8), 2112–2124. https://doi.org/10.1016/j.laa.2009.06.034

Pascual, Z. (2020). Una secuencia didáctica para la enseñanza de la transformación lineal: unificación de métodos y problemas, modelización y explicitación del aprendizaje. *Revista Latinoamericana de Investigación en Matemática Educativa*, *23*(3), 271–310. https://doi.org/10.12802/relime.20.2331

Pozzi, F., Manganello, F., & Persico, D. (2023). Collaborative learning: A design challenge for teachers. *Education Sciences, 13*(4), 331. https://doi.org/10.3390/educsci13040331

Rensaa, R., Hogstad, N., & Monaghan, J. (2020). Perspectives and reflections on teaching linear algebra. *Teaching Mathematics and its Applications: An International Journal of the IMA*, *39*(4), 296-309. https://doi.org/10.1093/teamat/hraa002

Rohrbeck, C. A., Ginsburg-Block, M. D., Fantuzzo, J. W., & Miller, T. R. (2003). Peer-assisted learning interventions with elementary school students: A meta-analytic review. *Journal of Educational Psychology*, *95*(2), 240–257. https://doi.org/10.1037/0022-0663.95.2.240

Roseth, C. J., Johnson, D. W., & Johnson, R. T. (2008). Promoting early adolescents' achievement and peer relationships: The effects of cooperative, competitive, and individualistic goal structures. *Psychological Bulletin*, *134*(2), 223–246. https://doi.org/10.1037/0033-2909.134.2.223

Stahl, G., Ludvigsen, S., Law, N. & Cress, U. (2014). CSCL artifacts. *International Journal of Computer-Supported Collaborative Learning*, *9*(3),327-245. https://doi.org/10.1007/s11412-014-9200-0

Strang, G. (2016). *Introduction to Linear Algebra*. Wellesley- Cambridge Press. https://math.mit.edu/~gs/linearalgebra/ila5/indexila5.html

Stuhlmann, A. S. (2019). Mathematics students talking past each other: emergence of ambiguities in linear algebra proof constructions involving the uniqueness quantification. *ZDM Mathematics Education*, *51*(7), 1083-1095. https://doi.org/10.1007/s11858-019-01099-9

Sutama, S., Narimo, H., Prayitno, J., Anif, D., & Sari, P. (2021). Mathematical collaborative learning in 21 st century based on national science Olympiad in junior high school. *Journal of Physics: Conference Series*, *1836*(1), 012046. https://doi.org/10.1088/1742-6596/1836/1/012046

Van de Pol, J., Volman, M., & Beishuizen, J. (2010). Scaffolding in teacher–student interaction: A decade of research. *Educational Psychology Review, 22*(3), 271-296. https://doi.org/10.1007/s10648-010-9127-6

Van de Pol, J., Volman, M., Oort, F., & Beishuizen, J. (2015). The effects of scaffolding in the classroom: Support contingency and student independent working time in relation to student achievement, task effort and appreciation of support. *Instructional Science*, *43*(5), 615-641. https://doi.org/10.1007/s11251-015-9351-z

Zariquiey, F. (2019). *Cooperar para aprender. Transformar el aula en una red de aprendizaje cooperative*. Ediciones SM. https://comunidadsm.com.pe/wp-content/uploads/2020/02/Cooperar-para-aprender.pdf