Journal on Mathematics Education
Volume 15, No. 4, 2024, pp. 1251-1276

Horizontal and vertical mathematization processes of 10th grade students: The case of Law of Sines

Quoc Ai Nguyen[1], Ngoc-Giang Nguyen[2,*]

[1]Faculty of Mathematics and Applications, SaiGon University, Ho Chi Minh City, Vietnam
[2]Faculty of Data Sciences in Business, Ho Chi Minh University of Banking, Ho Chi Minh City, Vietnam
*Correspondence: giangnn@hub.edu.vn

Received: 20 June 2024 | Revised: 5 October 2024 | Accepted: 15 November 2024 | Published Online: 16 November 2024

Abstract

Realistic Mathematics Education (RME) has gained significant attention in Vietnam over the past decade due to its potential for enhancing mathematics instruction. This study investigates the process of mathematicization undertaken by students as they transition from solving a real-world contextual problem to discovering and applying the Law of Sines. The primary problem involves determining the angle formed by the crossbar and the hanging rope of a disco ball. Guided by the principles of RME, the mathematicization process encourages students to model this scenario as a triangle with two given side lengths and a specified angle between one side and the base. With teacher facilitation, students construct a general mathematical model and subsequently reinvent the Law of Sines. They apply this law to solve the initial problem and further extend their understanding by tackling a similar contextual scenario. The study involved 40 students, who engaged with worksheets designed to present relevant problems. Their problem-solving processes were documented and analyzed using qualitative methods. The findings contribute to the development of a teaching approach for introducing the Law of Sines within the framework of RME, specifically tailored to the Vietnamese educational context. This approach underscores the progression of students' understanding through the integration of contextual problem-solving and theoretical reinvention.

Keywords: Law of Sines, Progressive Mathematization Process, Real Context Problem, Realistic Mathematics Education

How to Cite: Nguyen, Q. A., & Nguyen, N.-G. (2024). Horizontal and vertical mathematization processes of 10th grade students: The case of Law of Sines. *Journal on Mathematics Education, 15*(4), 1251–1276. https://doi.org/10.22342/jme.v15i4.pp1251-1276

Realistic Mathematics Education (RME) is an instructional approach in mathematics that emphasizes students' active engagement in constructing their understanding through problem-solving within meaningful, real-world contexts (Freudenthal, 2002). In this approach, teachers play a pivotal role in scaffolding students' learning, enabling them to reinvent mathematical concepts as they encounter them. A key feature of RME is the central role of "realistic" and contextually rich situations in the learning process. These contexts not only initiate the development of mathematical concepts, tools, and procedures but also serve as platforms where students can later apply their mathematical understanding (Hendroanto et al., 2018). Over time, this knowledge becomes increasingly formalized, generalized, and less tied to specific contexts (Van den Heuvel-Panhuizen & Drijvers, 2014; 2020).

Van den Heuvel-Panhuizen and Drijvers (2014; 2020) assert that the contemporary form of RME is largely shaped by Freudenthal's perspective on mathematics. He argued that mathematics should remain connected to reality, resonate with children's experiences, and be socially relevant to maintain its human value. Rather than viewing mathematics as a fixed body of knowledge to be transmitted, Freudenthal emphasized mathematics as an activity intrinsic to human experience. Mathematics education, therefore, should provide students with "guided" opportunities to "reinvent" mathematical concepts through active engagement. This perspective shifts the focus of mathematics education from treating mathematics as a static system to emphasizing the dynamic process of mathematization (Freudenthal, 1968).

Treffers (1991) identified five foundational principles of RME: creation and concretization, levels and models, in-depth thinking and specialized tasks, social context and interaction, and structuring through collaborative processing. These principles were later expanded by Van den Heuvel-Panhuizen (2000) and further elaborated in collaboration with Van den Heuvel-Panhuizen and Wijers (2005). In their work, these principles are categorized under six headings: the activity principle, the reality principle, the level principle, the intertwinement principle, the interaction principle, and the guidance principle (Inci et al., 2023; Risdiyanti & Prahmana, 2020; Van den Heuvel-Panhuizen, 2000).

The revised Mathematics General Education Curriculum in Vietnam (VNMGEC2018) emphasizes the application of RME theory in teaching mathematics, including the instruction of the Law of Sines. However, implementing RME in this context has revealed several challenges. Students often struggle to find solutions, understand the Law of Sines, and apply it in real-life situations. Furthermore, teachers lack sufficient materials and experience to effectively incorporate RME principles into lessons on this topic.

Applying the processes of horizontal and vertical mathematization in teaching the Law of Sines has demonstrated potential for improving instructional effectiveness. These processes encourage students to explore and construct knowledge independently while also recognizing the relevance of mathematics in real-world applications. Research on the implementation of horizontal and vertical mathematization in this context reveals inconsistencies in the effectiveness of various activities. These findings provide practical insights for educators to refine their lesson designs on the Law of Sines in Vietnam, promoting more engaging and effective mathematics instruction.

In Vietnam, various studies have explored aspects of RME theory, though their scope and focus differ. For instance, Da (2022) examined the application of RME principles to teaching calculus, while Danh Nam (2020) concentrated on the theoretical foundations of RME—such as modelling and the processes of horizontal and vertical mathematization—without providing practical examples. Similarly, Loc et al. (2020) discussed the application of RME in teaching the Law of Cosines, and Le et al. (2021) provided an overview of the benefits, challenges, and global debates surrounding RME. Additionally, research by Nguyen et al. (2020) proposed a framework to assess RME development in terms of policy and practice, offering insights and recommendations for its implementation in Vietnamese schools.

Despite these contributions, the application of the horizontal and vertical mathematization processes to teaching the Law of Sines remains an unexplored area. This represents a scientific gap in the research on RME within the Vietnamese context. The Law of Sines is a part of the 10th-grade Mathematics Curriculum, specifically within the Geometry and Measurement section, under Chapter 4 on Quantitative Relations in Triangles, in Lesson 2 alongside the Law of Cosines. The VNMGEC2018 emphasizes the importance of students not only explaining the Law of Sines but also applying it to solve practical problems. However, as highlighted by the VNMGEC2018, the Mathematics Curriculum tends to be logical, abstract, and generalized, which can pose challenges for learners. To address this, high school mathematics instruction must strike a balance between "learning" theoretical knowledge and "applying" it to concrete problem-solving scenarios (Vietnam Mathematics General Education Curriculum, 2018).

Given this context, this study focuses on exploring the horizontal and vertical mathematization processes in teaching the Law of Sines to 8th-grade students. This research direction aims to bridge the identified gap and enhance the practical application of RME in Vietnamese mathematics education.

Gravemeijer and Doorman (1999) define context problems as problem situations that are experientially real to students, aligning with Freudenthal's (2002) preference for using "reality" to describe what is perceived as real through common-sense experiences at a given stage. They highlight the reflexive relationship between context problems and the development of a student's experiential reality. On one hand, context problems are deeply rooted in the student's current reality; on the other hand, solving these problems expands and enriches this reality. Although the concept of "reality" within context problems is dynamic, instructional sequences often begin by connecting to students' everyday-life experiences.

In the reinvention approach central to RME, context problems play a pivotal role. Carefully selected context problems allow students to develop informal, context-specific solution strategies. These informal approaches can act as foundational inventions or serve as catalysts for processes such as curtailment, formalization, or generalization. Essentially, in RME, context problems are the cornerstone of progressive mathematization. Instructional designers aim to construct a series of context problems that facilitate horizontal and vertical mathematization, ultimately enabling students to reinvent the targeted mathematical concepts (Gravemeijer & Doorman, 1999).

Several researchers present a comprehensive rationale for incorporating contextual problems in mathematics education (Johnson et al., 2022; Blum & Niss, 1991; Boaler, 1993; Felton, 2010; Pierce & Stacey, 2006; Van den Heuvel-Panhuizen, 2005). They highlight seven theoretical arguments for this approach. Firstly, the formative argument emphasizes the role of contextual problems in developing students' broader competencies, attitudes, and skills, going beyond technical mathematical knowledge. Secondly, the critical competence argument underscores the importance of fostering mathematical literacy, equipping students with the ability to think critically and apply mathematical reasoning in everyday decision-making. The utility argument stresses the significance of contextual problems in enhancing knowledge transfer, enabling students to apply mathematical concepts in diverse settings, including future academic studies, real-life scenarios, and professional environments. The image of mathematics argument aims to enrich students' understanding of mathematics by presenting it as a dynamic, multifaceted, and intrinsically human activity, thereby countering the perception of mathematics as an isolated, rigid discipline. Additionally, the promotion of learning mathematics argument asserts that contextual problems provide relevance and motivation for students, aiding in the acquisition, retention, and application of mathematical concepts, methods, and results. The real-world application argument focuses on connecting mathematics to everyday life, demonstrating its relevance and practical value, and dismantling the perception of mathematics as an abstract, detached field. Finally, the halo effect argument suggests that teachers can use engaging and relatable contexts to improve students' attitudes toward mathematics, associating the subject with positive and enjoyable experiences.

Complementing these arguments, Gravemeijer (1994a) introduces three principles of realistic mathematical learning that guide instructional design. The first principle, guided reinvention and progressive mathematization, involves enabling students to rediscover mathematical concepts by transitioning from informal problem-solving strategies to formalized reasoning, ensuring gradual mastery of mathematical processes. The second principle, didactical phenomenology, involves analysing real-world phenomena to identify and develop meaningful mathematical concepts that are relevant to students' experiences. The third principle, self-developed models, encourages students to create their own representational tools to bridge the gap between real-life problems and abstract mathematical concepts, fostering a deeper understanding and ownership of their learning process.

Together, these arguments and principles advocate for embedding contextual problems in mathematics education. This approach not only enhances students' understanding and application of mathematical concepts but also fosters engagement, critical thinking, and the recognition of mathematics as an integral part of human activity and the real world.

Freudenthal (1968) conceptualized mathematics as the process of mathematizing reality, emphasizing that for most individuals, this process represents the ultimate purpose of mathematics. He argued that mathematics should not be taught as a static, closed system but rather as an active process involving the mathematization of reality and, where possible, the mathematization of mathematical concepts themselves. Freudenthal (1971) further expanded on this idea, defining "mathematizing" as a form of organizing that incorporates mathematical concepts. By using the term "organizing," he highlighted that mathematizing is not merely a translation into pre-established symbolic systems but a dynamic process through which methods of symbolization naturally emerge as one organizes subject matter (Gravemeijer & Terwel, 2000).

Freudenthal (1968) and Van den Heuvel-Panhuizen (2005) argued that just as mathematics originated from the mathematization of real-world phenomena, the learning of mathematics must similarly arise from engaging with and mathematizing reality. Early proponents of RME emphasized that if students learn mathematics in isolation, disconnected from their personal experiences, their understanding will be superficial, easily forgotten, and unlikely to be applied effectively. To counter this, mathematical learning should begin with rich, meaningful contexts that demand mathematical organization—contexts that inherently lend themselves to mathematization. Through engaging with such context problems, students can develop their mathematical tools and deepen their understanding.

Kant and Sarikaya (2021) identified two fundamental characteristics of mathematizing. First, symbolic notation plays a central role, enabling the representation of mathematical structures within a given context. At an introductory level, this involves using numbers, sketches, and symbols to capture and organize essential elements of a situation. For advanced mathematicians, symbolic notation serves as a critical tool for encoding and manipulating mathematical content. Second, the process of mathematization is inherently individualistic. It allows for diverse approaches and solutions, as each individual makes unique choices in how they organize and represent mathematical relationships. In summary, mathematizing, as conceptualized by Freudenthal and further elaborated by subsequent researchers, is an active, individual process rooted in real-world contexts. It involves both the use of symbolic notation and the creative organization of mathematical ideas, providing a foundation for meaningful and enduring mathematical learning.

The process of mathematization consists of several structured steps aimed at bridging real-world problems and mathematical understanding. According to De Lange (2006), the process begins by identifying a real-world problem and then reorganizing it based on relevant mathematical concepts. This is followed by a gradual abstraction, or elimination of reality, leading to the formulation and solution of a strictly mathematical problem. Finally, the mathematical solution is interpreted in the context of the original real-world problem. The first three steps—starting with a problem, identifying relevant mathematical concepts, and abstracting from reality—guide learners from the real-world context into the realm of mathematical analysis.

Expanding on Freudenthal's concept of mathematizing, Treffers (1987) introduced a distinction between two forms: horizontal and vertical mathematization. Horizontal mathematization involves translating a contextual problem into a mathematical problem, effectively moving from the real world to the symbolic mathematical domain. In contrast, vertical mathematization focuses on restructuring and advancing mathematical knowledge, elevating mathematical reasoning to a higher conceptual level—what Treffers (1987) referred to as "mathematizing mathematics." Gravemeijer and Terwel (2000) suggest that vertical mathematization can be fostered by presenting problems that allow solutions across different levels of mathematical abstraction.

Freudenthal (2002) adopted and refined Treffers (1987) distinction, emphasizing the complementary nature of horizontal and vertical mathematization. In horizontal mathematization, students employ mathematical tools to analyse and solve real-world problems, moving from practical situations into the abstract world of mathematical symbols. Vertical mathematization, on the other hand, involves creating connections and discovering shortcuts within the mathematical system, deepening understanding and efficiency in problem-solving. It is characterized by progression within the symbolic domain, leveraging relationships between concepts and strategies.

Van den Heuvel-Panhuizen and Drijvers (2014; 2020) stress that both forms of mathematization are equally important and interdependent. While RME's focus on real-world contexts often emphasizes horizontal mathematization, excessive attention to real-life applications may risk undervaluing the role of vertical mathematization. Balancing both aspects ensures that students not only connect mathematics to their lived experiences but also develop the abstract reasoning skills essential for deeper mathematical understanding.

In the process of progressive mathematization, learners engage with mathematical concepts through two distinct stages known as horizontal and vertical mathematization (Treffers, 1987). During horizontal mathematization, learners tackle contextual tasks by employing informal strategies to describe and solve problems grounded in real-life situations. Starting from a specific context, they model the task by identifying relationships and patterns, which may result in mathematical structures such as tables, graphs, or formulas (Nkambule, 2009). Horizontal mathematization involves reformulating problem situations to make them suitable for mathematical analysis. This process includes a variety of exploratory activities, such as experimenting, identifying patterns, classifying, conjecturing, and organizing data, which serve as the foundation for further mathematical reasoning.

Vertical mathematization, on the other hand, builds upon the groundwork laid by horizontal mathematization. It involves reasoning about abstract structures, generalizing findings, and formalizing concepts. Through these activities, learners create new mathematical realities that can subsequently become the context for further mathematization—whether horizontal or vertical—forming a continuous sequence of progressive mathematizations. In this way, vertical mathematization enables learners to construct increasingly sophisticated mathematical knowledge, with their evolving understanding serving as the basis for subsequent exploration and abstraction (Rasmussen et al., 2005).

The principle of guided reinvention, central to RME, focuses on the learning process by encouraging students to view the knowledge they acquire as their own. The intent is to make learners feel responsible for their understanding, as they actively construct their knowledge through exploration and discovery. For educators, this implies providing opportunities for students to develop their own mathematical frameworks based on meaningful engagement with problems. Historical developments in mathematics serve as a heuristic tool in this process, demonstrating how informal solution strategies can anticipate more formal procedures. In this way, students' informal strategies are interpreted as stepping stones toward formalization, with the reinvention process emerging naturally through contextual problems that encourage diverse solution approaches.

Freudenthal (2002) and subsequent scholars (Gravemeijer, 1994b; Gravemeijer & Terwel, 2000; Streefland, 1991) highlight the importance of selecting contextual problems that allow for multiple solution strategies, ideally reflecting potential learning trajectories. Freudenthal underscores the term "guided reinvention" to emphasize the balance between discovery and instruction. While learners are encouraged to "discover" knowledge independently, the learning environment is structured to guide their exploration. The "reinvention" process refers to steps in the learning journey where learners organize and internalize mathematical knowledge, while "guided" points to the teacher's role in shaping the instructional setting. This dynamic allows students to construct their mathematical understanding in a way that is both personally meaningful and pedagogically effective.

Guided reinvention serves as a cornerstone for instructional design within RME, functioning as a mission statement for RME-based design studies. Gravemeijer (1999) and Larsen (2018) emphasize that guided reinvention is not merely about motivating students with real-life contexts but about identifying contexts that feel experientially real and meaningful to the students. This principle encourages instructional designers to create learning environments where students construct their own mathematical understanding, grounded in contexts they can relate to and build upon.

A critical element of this process is the role of models in bridging informal and formal mathematical reasoning. Streefland (1985) describes the transition from a "model of" to a "model for" as a key component of this progression. Initially, models are closely tied to specific problem contexts, helping students make sense of the situation. Over time, these models are generalized, evolving into tools that students can use to organize new and related problem situations and to reason mathematically at a more formal level. At this stage, problem-solving strategies become less context-specific, reflecting a broader and more abstract mathematical perspective (Van Den Heuvel-Panhuizen, 2003).

Gravemeijer (1994a) highlights the instructional goal of enabling learners to develop self-generated models that evolve into more formal mathematical frameworks. Through real-world tasks and collaborative discussions, students create models that initially reflect the real-world situation but gradually transform into abstract tools for mathematical reasoning (Yackel et al., 2003). Larsen (2018) elaborates on this progression through the emergent models heuristic, which conceptualizes how students' informal mathematical activities, rooted in contextual starting points, can develop into the formal mathematics targeted by instruction. This process, known as the model-of/model-for transition, begins with the concept serving as a model for the teacher's observation of student activity. As students gain deeper understanding, the concept becomes a "model for" their own use, guiding future mathematical activity.

The principle of didactical phenomenology, introduced by Freudenthal (1983), further supports the design of instructional sequences. Freudenthal defines mathematical concepts as tools developed to organize phenomena in the physical, social, and mental worlds. Didactical phenomenology involves describing mathematical ideas in relation to the phenomena they were created to address and to the extensions they have gained throughout human learning. From a pedagogical perspective, this approach helps teachers identify points where learners can naturally enter the ongoing mathematical learning process. Freudenthal advises focusing on living roots of mathematical knowledge while avoiding unproductive historical complexities.

Larsen (2018) explains that the "thought-matter" in this context refers to the mathematical concepts students are meant to learn, and "organizing" involves mathematizing the context. Didactical phenomenology, therefore, advises designers to embed learning in contexts that can be effectively organized by the mathematics being taught. Gravemeijer and Terwel (2000) emphasize this principle by advocating for phenomenologically rich situations—contexts that naturally lend themselves to being organized by the mathematical objects students are constructing. Unlike concept-attainment approaches that rely on concrete materials to embody abstract ideas, Freudenthal proposed selecting situations that inherently "beg to be organized" mathematically. This approach ensures that students' mathematical activity is deeply rooted in meaningful, context-driven exploration, fostering both conceptual understanding and abstraction.

Together, guided reinvention, emergent models, and didactical phenomenology form a cohesive framework for designing instructional sequences in RME, promoting a seamless progression from informal problem-solving to formal mathematical reasoning. These principles ensure that learning is grounded in students' experiences while encouraging the development of abstract mathematical tools for broader application.

METHODS

The study began with the development of a mathematization process, a teaching model for the Law of Sines aligned with the principles of RME, and a structured teaching process for implementing the Law of Sines. This included selecting appropriate samples and data, as well as determining suitable data analysis techniques. Subsequently, an experimental teaching session was conducted with a group of 40 tenth-grade students from Class 10C4 at the High School of Gifted Sports in Binh Chanh District, Ho Chi Minh City.

The students were divided into 10 groups, each comprising 4 members with varying academic abilities categorized as good, fair, satisfactory, and unsatisfactory. This grouping aimed to foster collaborative learning and ensure a balanced distribution of skills within each group. The groups were tasked with rediscovering the Law of Sines following the RME approach. To initiate the learning process, a real-life problem was introduced to provoke curiosity and active participation. The problem presented a scenario in which students were required to determine an unknown angle given the measurements of two sides and an angle that was not included between them.

Students engaged in self-guided exploration, leveraging their prior knowledge to attempt solutions to the problem. The teacher provided corrective feedback on their approaches and facilitated the discovery of the Law of Sines by guiding students through their thought processes. Once students had identified the Law of Sines, they applied it to solve the original problem. To deepen their understanding, additional real-life problems relevant to Vietnam were provided, requiring students to utilize the Law of Sines for practical applications.

To promote active participation, representatives from selected groups presented their solutions to the entire class. Other groups were encouraged to comment on and evaluate these solutions, fostering peer learning and critical thinking. The teacher consolidated the results, offered feedback on the learning process, and assessed each group's participation, collaboration, and progress. A combination of group and individual analysis techniques was employed to ensure a comprehensive evaluation of the students' performance and engagement throughout the activity.

Designing the Mathematization Process

De Lange (2006) outlined a cyclic process of mathematization that consists of five interrelated steps, such as read and understand the problem in a real-life context, identify related mathematical knowledge and reorganize the problem according to identified mathematical concepts, gradually eliminate the reality of the problem and turn the real - life problem into a mathematical problem, solve problems using mathematical tools, and check the reasonableness of the problem solution in step (4) with the real world, as illustrated in Figure 1.

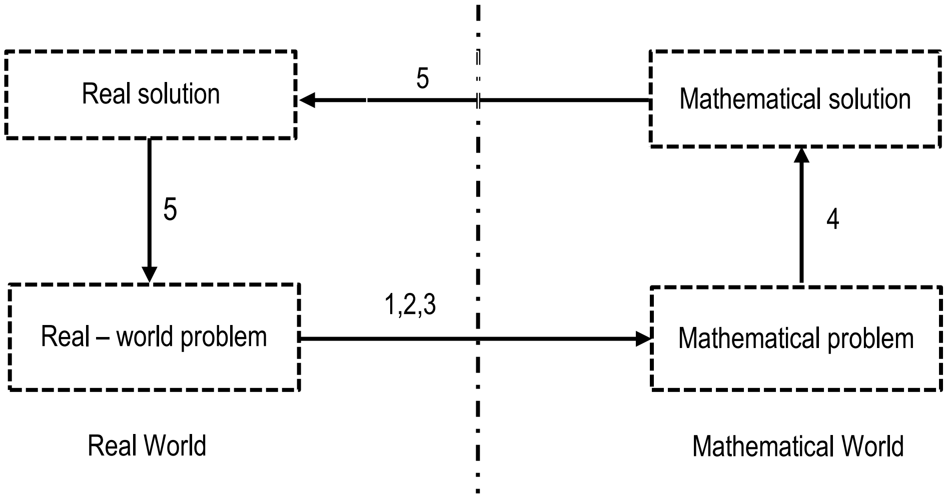


Figure 1. The mathematisation cycle (De Lange, 2006; Jupri & Drijvers, 2016)

However, we concur with the mathematization process outlined by Jupri and Drijvers (2016), which is based on the framework proposed by De Lange (2006). This process involves several key steps: beginning with a problem rooted in real-world contexts, the mathematization process starts with understanding the problem to identify relevant mathematical concepts embedded within it. Subsequently, the problem is formulated into a mathematical model, followed by solving the mathematical problem. The process concludes with a reflection on the solution and its interpretation, as illustrated in Figure 2.

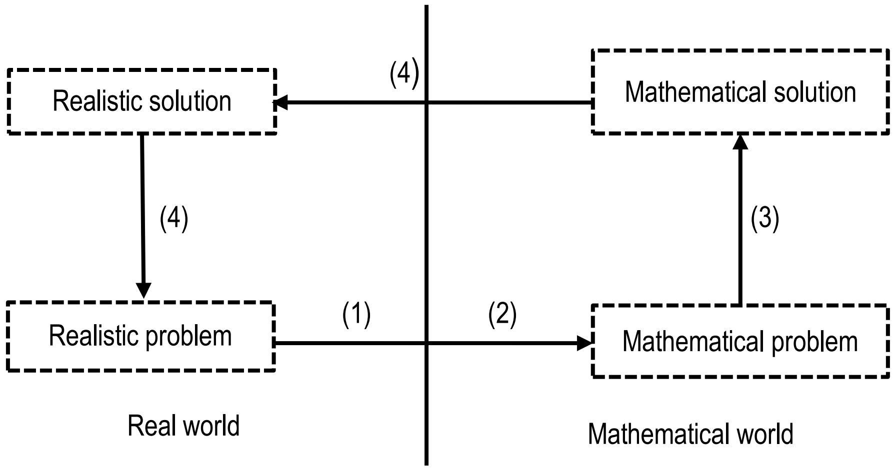


Figure 2. The mathematization cycle (De Lange, 2006; Jupri & Drijvers, 2016)

Designing the Teaching Model of the Law of Sines according to RME

The design of the teaching model is grounded in the theoretical frameworks of didactical phenomenology, emergent models heuristics, and guided reinvention. However, particular attention must be given to the lesson planning framework in Vietnam to effectively integrate RME principles. This alignment is crucial for creating favorable conditions that facilitate the application of RME in teaching mathematical topics in Vietnam, thereby demonstrating the feasibility and necessity of further developing RME in this context.

According to the Lesson Planning Framework issued by the Ministry of Education and Training of Vietnam (MOEVN) in December 2020, the teaching model for a mathematics topic consists of four key activities. The first activity involves identifying the problem, learning task, or introduction. This step is designed to help students recognize the specific problem or task to be addressed in the lesson, or to clearly define the approach for solving the problem or performing the task in subsequent activities. The second activity focuses on forming new knowledge, solving problems, or performing tasks outlined in the first activity. Here, students engage in learning tasks to acquire new knowledge or address the problems introduced earlier.

The third activity is practice, aimed at enabling students to apply their acquired knowledge while fostering the development of application skills. Finally, the fourth activity emphasizes application, where students are encouraged to apply their knowledge and skills to practical tasks or scenarios. This activity supports the development of student competencies and can be tailored to individual lessons or groups of lessons with appropriate content.

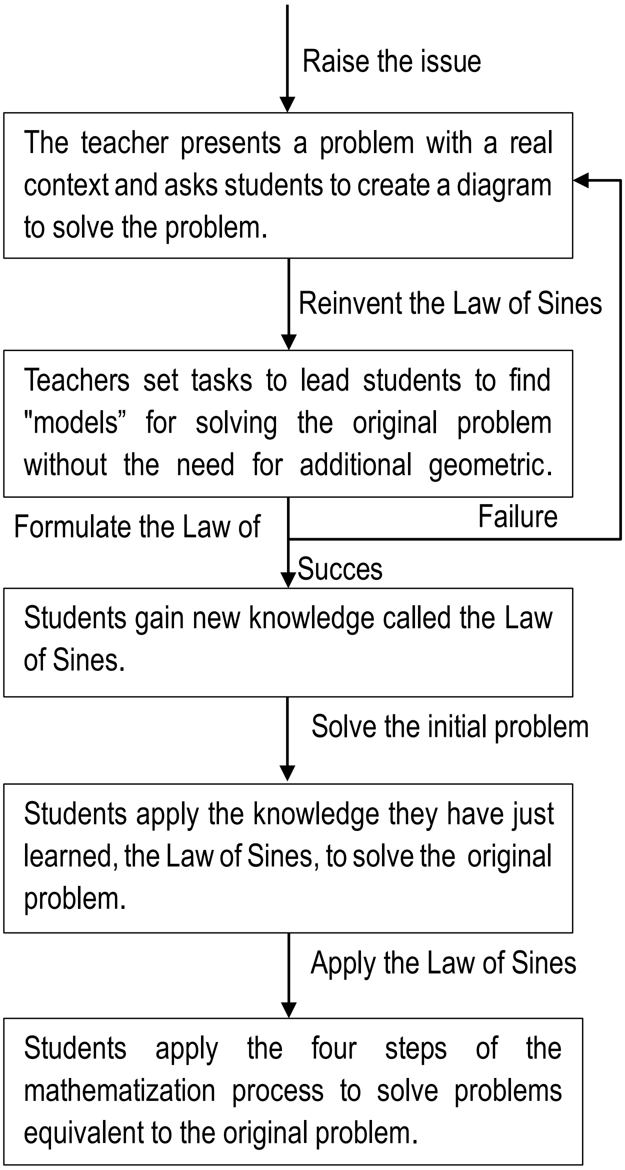


Figure 3. The teaching model of the law of sines according to RME

Based on the Lesson Planning Framework issued by the Ministry of Education and Training of Vietnam (MOEVN), we designed a teaching model for the topic *The Law of Sines*. This model is structured into five distinct activities, ensuring comprehensive alignment with the educational goals as illustrated in Figure 3. The activities are as follows:

1. Raise the Issue: In this activity, the teacher introduces a problem rooted in a real-world context. Students are tasked with creating a diagram to represent the problem by identifying its requirements, translating the contextual problem into a mathematical model, and using a combination of existing mathematical tools to propose solutions.
2. Reinvent the Law of Sines: Teachers guide students through tasks designed to help them discover "models" that can solve not only the original problem but also similar problems without the need for additional geometric constructions.
3. Formulate the Law of Sines: At this stage, students acquire new knowledge, which is formally presented as the Law of Sines.
4. Solve the Initial Problem: Students apply their newly acquired knowledge of the Law of Sines to solve the original problem introduced in the first activity.
5. Apply the Law of Sines: In this activity, the teacher presents a new problem in a different context. Students are required to apply the four steps of the mathematization process (as outlined in Figure 3) to solve this new problem, thereby reinforcing their understanding and application of the Law of Sines.

This structured approach ensures that students engage in a progressive learning process, from problem identification and knowledge discovery to application and reflection, aligning with both theoretical and practical learning objectives.

Teaching Process of the Law of Sines according to RME

The above mathematization process is integrated across the five activities of the teaching model as follows:

1. Activity 1: Raise the Issue
   In this activity, students engage in Step 1 of the mathematization process. When the teacher presents a real-world problem, students begin by reading, understanding, and identifying the requirements of the problem. For instance, they are tasked with determining the angle formed by a chain and a horizontal bar. Subsequently, students proceed to Steps 2 and 3, where they select relevant mathematical concepts from prior knowledge, develop a mathematical "model" of the problem, and present a solution to this model.
2. Activity 2: Reinvent the Law of Sines
   In this activity, students continue with Steps 2 and 3 of the mathematization process. They construct a generalized "model" for a class of problems equivalent to the initial problem. This involves seeking a more efficient solution to the model, focusing on directly using the given data without introducing additional geometrical elements. Through this exploration, students independently derive the Law of Sines.
3. Activity 3: Formulate the Law of Sines
   Teachers assist students in formalizing their discovery of the Law of Sines. Students engage in Step 3 of the mathematization process by proving the validity of the Law of Sines for right-angled and obtuse triangles, consolidating their new understanding through logical reasoning and

validation.

4. Activity 4: Solve the Initial Problem
   Students return to the original real-world problem posed in Activity 1, now equipped with the Law of Sines as a mathematical tool. Here, they perform Steps 3 and 4 of the mathematization process. They use the Law of Sines to solve the mathematical model of the problem and assess whether their solution aligns with the practical context, thereby ensuring the result is both mathematically accurate and contextually relevant.
5. Activity 5: Apply the Law of Sines
   In this activity, the teacher presents a new problem within a different practical context. Students are required to apply all four steps of the mathematization process. They begin by reading and understanding the new problem (Step 1) and identifying the necessary data to construct a mathematical "model" of the problem, typically a triangle (Step 2). Students then apply the Law of Sines to solve the model (Step 3) and evaluate the reasonableness of their results in relation to the original practical situation (Step 4).

This structured approach ensures that students not only understand the theoretical foundation of the Law of Sines but also develop the ability to apply it effectively in diverse practical contexts, fostering deep conceptual understanding and problem-solving skills.

RESULTS AND DISCUSSION

The Process of Horizontal and Vertical Mathematization

Table 1 presents the outcomes of horizontal and vertical mathematization observed throughout all activities conducted during the learning process. The details are summarized as follows:

1. Activity 1: Raising the Issue
   In this phase, all ten groups (comprising 40 students) successfully engaged in both horizontal and vertical mathematization. They constructed triangle ABC as a "model-of" the problem and utilized previously acquired mathematical knowledge to solve it effectively.
2. Activity 2: Reinventing the Law of Sines
   Nine groups (36 students) successfully performed vertical mathematization to address the general model derived from the initial "model-of" the problem. These groups successfully developed the Law of Sines (specific to acute triangles) as a "model-for" the generalized problem. However, one group (4 students) encountered errors when dividing fractions during the vertical mathematization process, which impeded their success in this activity.
3. Activity 3: Formulating the Law of Sines
   All students demonstrated success in vertical mathematization during this activity. All ten groups succeeded in formulating the Law of Sines and provided proofs verifying its applicability to both right and obtuse triangles.
4. Activity 4: Solving the Initial Problem
   All participants achieved successful horizontal and vertical mathematization in this phase. They effectively applied the Law of Sines to solve the initial problem.
5. Activity 5: Applying the Law of Sines
   In this activity, all groups successfully performed horizontal mathematization by constructing two right triangles with a shared right angle as a "model-of" the third problem. Nevertheless, during the

application of the Law of Sines to solve this "model-of" problem, one group faced challenges and was unable to complete the solution. Consequently, nine groups successfully implemented vertical mathematization in this activity.

Table 1. Results on horizontal and vertical mathematization

Activities	Horizontal Mathematization		Vertical Mathematization	
	Successful	**Unsuccessful**	**Successful**	**Unsuccessful**
Raise the issue	10 groups	0	10 groups	0
Reinvent the Law of Sines			9 groups	1 group
Formulate the Law of Sines			10 groups	0
Solve the initial problem	10 groups	0	10 groups	0
Apply the Law of Sines	9 groups	1 group	9 groups	1 group

The class was divided into 10 groups, each comprising students with varying levels of learning abilities. As a result, the initial activity, Raise the Issue, achieved a 100% success rate. However, during the subsequent activity, Reinvent the Law of Sines, one group failed to establish the relationship between the sides and angles due to calculation errors. Despite this, all groups successfully completed the subsequent steps, Formulate the Law of Sines and Solve the Initial Problem. This outcome can be attributed to the relatively strong foundational knowledge, skills, and problem-solving abilities of students in Vietnam. Nevertheless, during the final step, Apply the Law of Sines, one group encountered difficulties in calculating the required angle, which prevented them from effectively applying the law in practical contexts.

In terms of the horizontal and vertical mathematization processes, the activities Raise the Issue and Reinvent the Law of Sines fall under horizontal mathematization. Conversely, the activities Formulate the Law of Sines and Solve the Initial Problem align with vertical mathematization. The final step, Apply the Law of Sines, integrates both horizontal and vertical mathematization.

Table 2 summarizes the success rates across the three types of mathematization. The analysis of Table 2 highlights that vertical mathematization yielded the highest success rate at 100%. This outcome suggests that students demonstrated strong reasoning and procedural skills when abstractly formulating and solving mathematical problems. In contrast, horizontal mathematization had a slightly lower success rate of 90%, reflecting the challenges some groups faced when exploring and contextualizing real-world scenarios, as seen in Reinvent the Law of Sines. The combined horizontal and vertical mathematization also had a success rate of 90%, indicating that integrating both processes introduced additional complexities, as illustrated by the challenges in Apply the Law of Sines. These findings emphasize the need for targeted interventions to improve students' abilities in applying theoretical knowledge to practical contexts.

Table 2. The comparison of vertical and horizontal mathematization

Mathematization	Horizontal mathematization	Vertical mathematization	Horizontal and vertical mathematization
Success	90%	100%	90%

Teaching Process

Activity 1. Raise the Issue

The teacher gives the groups worksheet 1 including the following real - life problem, as follow,

At a bar, a disco ball is hung from the crossbar by two cables (Figure 4). The first cable is 4.3 m long and forms a 65° angle with the crossbar. The remaining cable is 5.4 m long. How many degrees does the second cable make with the crossbar? (round result to degrees)

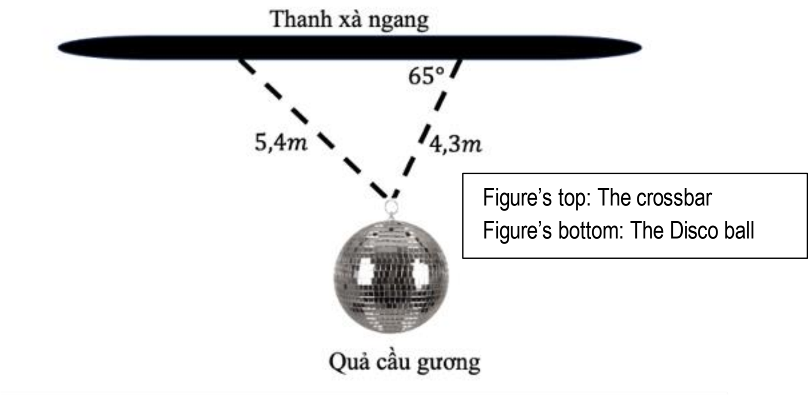


Figure 4. The disco ball hangs under the crossbar

After reading the problem, groups actively discuss the requirements of the problem. The teacher asks the students to create a diagram of the steps to solve the problem. After several minutes of discussion, the results show that all groups were able to create a diagram of the steps to solve the problem on the poster.

All student groups use inverse inference to guide their solutions. They construct the altitude *CH* perpendicular to *AB* at *H* to get 2 right triangles: triangle *ACH* right at *H* and triangle *BCH* right at *H*. Consider triangle *ACH* is right at *H* with side length *AC*, so to calculate the angle $\hat{A}$ requires finding the length of side *CH*. To find the length of side *CH*, use $sin\hat{B} = CH/BC$ in triangle *BCH* right at *H*. Finally use $sin\hat{A} = CH/AC$ find the measure of angle $\hat{A}$.

Teacher calls group 1 to post their group's poster on the board. Figure 5 is a diagram of group 1's solving process.

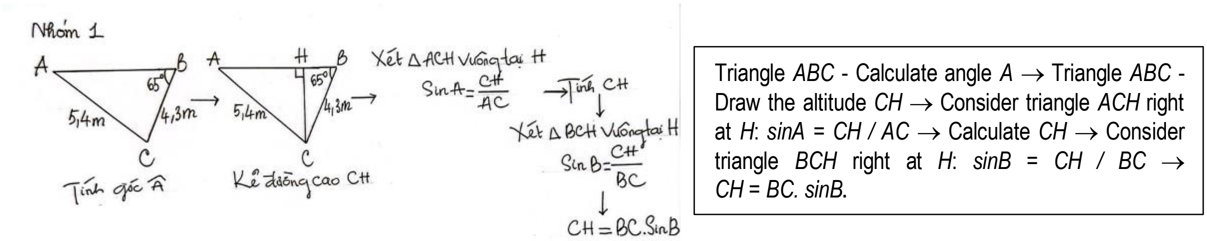


Figure 5. Diagram of problem-solving steps for Group 1

The teacher comments on group 1's work: "Group 1 recognizes the disco ball, together with the two cables and the crossbar, form a triangle. The group calls the two vertices of the triangle on the crossbar from left to right vertex A and vertex B, respectively, and the remaining vertex at the top of the disco ball is C.

The cable on the right is $4.3\ m$ long, so $BC = 4.3\ m$ and this cable create an angle of $65°$ with the crossbar, so $\hat{B} = 65^0$. The cable on the left is $5.4\ m$ long, so $AC = 5.4\ m$. To know how many degrees the left cable makes with the crossbar, that is, to find the measure of angle $\hat{A}$, the group constructs the altitude CH to form a triangle ACH right-angled at H and use $sin\hat{A} = CH/AC$. The group knows how to use $sin\hat{B} = CH/BC$ or $CH = BC.sin\hat{B}$ in the triangle BCH right-angled at H to find CH. The work of group 1 shows that to calculate the measure of angle $\hat{A}$, one needs to know the length of sides AC and HC. On the other hand, to calculate the length of side HC, one needs to know the measure of angle $\hat{B}$ and the length of side BC. So, to calculate the measure of angle $\hat{A}$, it all boils down to the lengths of sides AC, BC, and the measure of angle $\hat{B}$.

In this activity, students analyze the problem and determine that the problem's requirement is to find the angle created by the left cable with the crossbar. After selecting the necessary information including the lengths of the two cables which are $5.4\ m$ and $4.3\ m$, and the cable on the right making an angle of 65^0 with the crossbar, they name the vertices and construct model (triangle ABC) for real - life problem. At that time, horizontal mathematization is demonstrated through students transferring the problem in a real context to a triangular model with accompanying data. Then, students use the trigonometric ratio of the acute angles $sin\hat{B} = CH/BC$ in triangle BCH right-angled at H and $sin\hat{A} = CH/AC$ in triangle ACH right-angled at H to solve the problem. Thus, in this activity 1, all groups construct a "model - of" the original problem, which is triangle *ABC* with altitude CH, and two relations $sin\hat{A} = CH/AC$ and $sin\hat{B} = CH/BC$

Activity 2. Reinvent the Law of Sines

The teacher poses the problem: The work of all groups shows that calculating the measure of angle $\hat{A}$ all boils down to constructing the altitude *CH* and then calculating its length through BC and $sin\hat{B}$. So, is there any other way to calculate the measure of angle $\hat{A}$ in terms of BC, CA and $\hat{B}$ more quickly? More specifically, find the relation between the measure of angle $\hat{A}$, the length of side BC, the measure of angle $\hat{B}$, and the length of side AC.

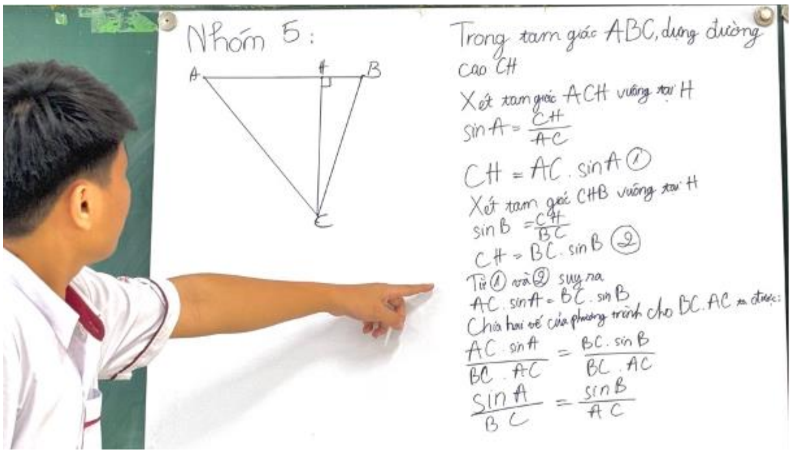
Teachers continue to give the following tasks:

1. After drawing the altitude CH, using the two newly formed right triangles, write the trigonometric ratio of $sin\hat{A}$, do the same with $sin\hat{B}$.
2. From the above two relations, find a relation that does not depend on the quantity CH.
3. Write the found relation as the ratio of the sine value of an angle to the length of the side opposite that angle.

Students actively work in groups to perform assigned tasks.

The results of the groups' work show that: 9/10 groups find the relation between $sin\hat{A}$, the length of side BC, $sin\hat{B}$, the length of side AC (part of the Law of Sines). Particularly group 4 has difficulty finding the relation. At the same time, the teacher calls group 4, the unsuccessful group, and group 5, the successful group, to present their group's solutions on the board.

Teacher comments: Group 4 correctly writes 2 trigonometric ratios $sin\hat{A} = CH/AC$ and $sin\hat{B} = CH/BC$. To find a relation that does not contain CH, the group divides the two found relations side by side. However, the group makes a mistake of $(CH/AC)/(CH/BC) = AC/BC$ by not applying the rule for dividing two fractions correctly. Regarding group 5's work illustrated in Figure 6, after correctly writing the two trigonometric ratios $sin\hat{A}$ and $sin\hat{B}$, the group realizes that these two equations had the same CH, so they calculate CH in terms of AC and $sin\hat{A}$ as well as BC and $sin\hat{B}$. Then, they derive the relation $AC.sin\hat{A} = BC.sin\hat{B}$, then divide both sides by $BC.AC$ to get the relation required by the teacher.



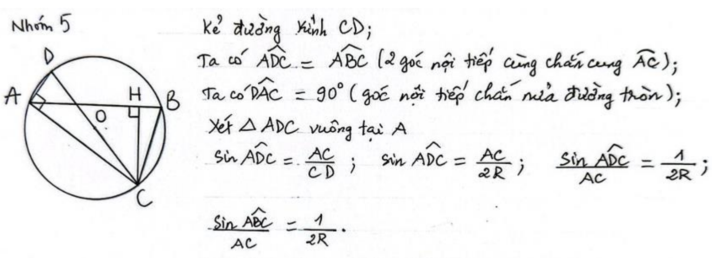
Group 5: In triangle *ABC*, construct altitude *CH*.
Consider triangle *ACH* right at *H*:
sinA = *CH* / *AC*; *CH* = *AC.sinA* (1)
Consider triangle *CHB* right at *H*:
sinB = *CH* / *BC*
CH = *BC.sinB* (2)
From (1) and (2), we deduce that
AC.sinA = *BC.sinB*
Divide both sides of the equation by *BC.AC*, we get:

$$\frac{AC.sinA}{BC.AC} = \frac{BC.sinB}{BC.AC}$$

$$\frac{sinA}{BC} = \frac{sinB}{AC}.$$

Figure 6. Group 5's solution

The teacher continued to give the task: draw a circle passing through the three vertices A, B, C of triangle and prove that $sin\hat{B}/AC = 1/2R$ with *R* being the radius of the circle circumscribing the triangle *ABC*. The results showed that all groups were able to draw the circumcircle of triangle *ABC* and prove $sin\hat{B}/AC = 1/2R$ by constructing the diameter *CD*, using trigonometric ratio $sin\widehat{ADC} = AC/CD$ in triangle *ADC* right-angled at *A*, and then use the result $sin\widehat{ADC} = sin\widehat{ABC}$ because these 2 angles are inscribed angles that intercept the arc $\overset{\frown}{AC}$. The teacher calls group 5 to continue presenting their group's work on the board, as shown in Figure 7.



Group 5: Draw diameter *CD*
We have $\widehat{ADC} = \widehat{ABC}$ (Two inscribed angles intercept arc *BC*);
We have $\widehat{DAC} = 90^0$ (inscribed angle intercepts semicircle);
Consider triangle *ADC* right at *A*
$sin\widehat{ADC} = AC/CD$; $sin\widehat{ADC} = AC/2R$;
$\frac{sin\widehat{ADC}}{AC} = \frac{1}{2R}$; $\frac{sin\widehat{ABC}}{AC} = \frac{1}{2R}$.

Figure 7. Group 5's solution

Representative of group 5 explains: "To prove that $sin\hat{B}/AC = 1/2R$, we try to make R appear by drawing a diameter from vertex A, B, or C. The question is whether we will draw the diameter from which vertex? Angle $\widehat{ABC}$ intercepts arc $\overset{\frown}{AC}$ so we will draw a diameter from vertex C to form angle $\widehat{ADC}$ equal to angle $\widehat{ABC}$ then use trigonometric ratios $sin\widehat{ADC} = AC/CD$ in the triangle *ADC* right-angled at $\hat{A}$. Substituting $CD = 2R$ and $sin\widehat{ADC} = sin\widehat{ABC}$ into the last relation, we get the relation needed to be proved.

In this activity, from the mathematical model of triangle ABC with altitude CH, students can determine the mathematical tools such as the trigonometric ratios of acute angles $sinA = CH/AC$ and $sin\hat{B} = CH/BC$ respectively in the triangle ACH right-angled at H and CHB right-angled at H, transitive properties, proportional properties to find the "model – for" being part of the Law of Sines for acute triangles. In addition, they also know how to use the property that inscribed angles that intercept the same arc are equal to transform angle $\widehat{ADC}$ to angle $\widehat{ABC}$ or that an inscribed angle that intercepts a half circle is a right angle, so triangle ADC is right at A to prove that $sin\hat{B}/AC = 1/2R$. The students' combined use of mathematical tools shows that a vertical form of mathematization is demonstrated.

Activity 3. Formulate the Law of Sines

From finding a solution for the mathematical model constructed above, the teacher leads the students to form the Law of Sines with a series of questions.

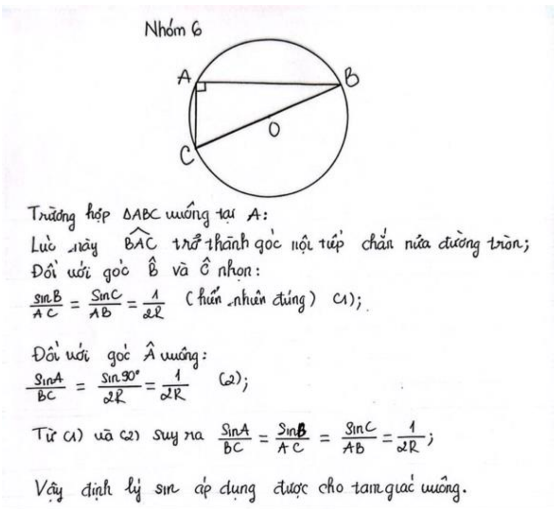
> *Teacher: What do you think about $sinA/BC$; $sinB/AC$ and $1/2R$?*
> *Group 3 volunteers to answer: $sinA/BC$ = $sinB/AC$ = $1/2R$.*
> *Teacher: By doing the same for angle $\widehat{ACB}$, what result will we get??*
> *Group 3: Yes, we also have $sinC/AB = 1/2R$.*
> *Teacher: What do you guys think about $sinA/BC$; $sinB/AC$; $sinC/AB$ and $1/2R$?*
> *Group 3 said: Yes, we find: $sinA/BC$ = $sinB/AC$ = $sinC/AB$ = $1/2R$.*

The teacher introduces to students the found relation called the Law of Sines, and the teacher asks students to verbally state the law for arbitrary triangle ABC.

Group 9: In any triangle ABC, the ratios between the sine value of an angle and the length of the opposite side are equal and equal to $1/2R$. After that, the teacher institutionalizes the Law of Sines: In triangle ABC, we have: $sinA/BC = sinB/AC = sinC/AB = 1/2R$, where R is the radius of the circumcircle of triangle ABC.

Thus, in this activity 3, students rediscovered the "model – for" acute triangles called the Law of Sines under the guidance of the teacher. The teacher asks other question: Does the Law of Sines apply to right triangles and obtuse triangles? Explain?

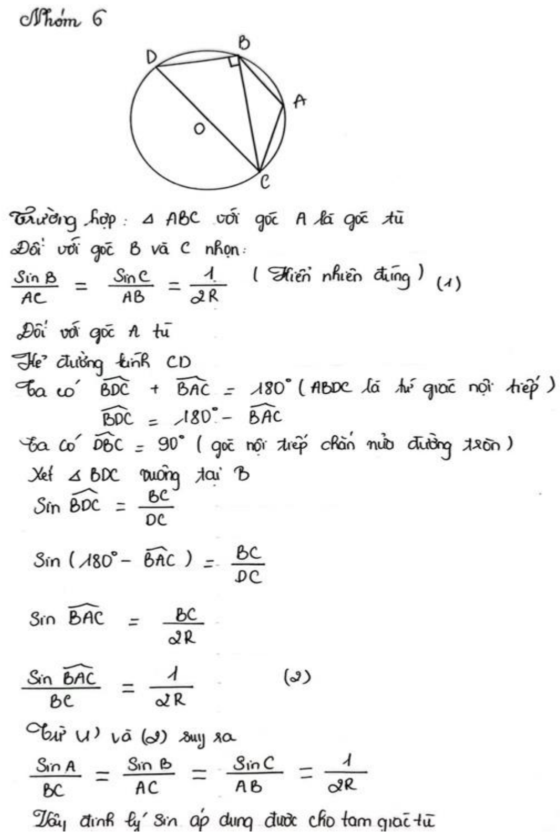
The results showed that all groups provided reasonable arguments to answer this question. The teacher called group 6 to present their answer (Figures 8 and 9). The remaining groups discuss and agree with group 6's answer. Teacher concludes that the presentation of group 6 was reasonable.



Group 6:
In case triangle *ABC* is right at *A*:
At this time, angle $\widehat{BAC}$ becomes an inscribed angle that intercepts the semicircle;
For acute angles $\hat{B}$ and $\hat{C}$:
$\frac{sinB}{AC} = \frac{sinC}{AB} = \frac{1}{2R}$ (obviously true) (1);
For right angle $\hat{A}$:
$\frac{sinA}{BC} = \frac{sin90^0}{2R} = \frac{1}{2R}$ (2);
From (1) and (2) we can deduce that: $\frac{sinA}{BC} = \frac{sinB}{AC} = \frac{sinC}{AB} = \frac{1}{2R}$;
So, the Law of Sines can be applied to right triangles.

Figure 8. Group 6's presentation that the triangle ABC is a right triangle

In this activity, in the case of right triangle ABC at A, students use the following mathematical rules to deduce the Law of Sines: $sin90° = 1$, transitive property, inscribed angle intercepts the semicircle is a right angle.



Group 6
In the case of triangle *ABC* with obtuse angle *A*
For acute angles *B* and *C*:
$\frac{SinB}{AC} = \frac{SinC}{AB} = \frac{1}{2R}$ (obviously true) (1)
For obtuse angle *A*
We draw diameter *CD*
We have $\widehat{BDC} + \widehat{BAC} = 180°$ (*ABCD* is a quadrilateral inscribed in a circle)
$\widehat{BDC} = 180° - \widehat{BAC}$
We have $\widehat{DBC} = 90°$ (inscribed angle intercepts semicircle)
Consider triangle *BDC* right at *B*

$$Sin\widehat{BDC} = \frac{BC}{DC}$$

$$Sin\left(180° - \widehat{BAC}\right) = \frac{BC}{DC}$$

$$Sin\widehat{BAC} = \frac{BC}{2R}$$

$$\frac{Sin\widehat{BAC}}{BC} = \frac{1}{2R} \quad (2)$$

From the two results (1) and (2), we can deduce that

$$\frac{SinA}{BC} = \frac{SinB}{AC} = \frac{SinC}{AB} = \frac{1}{2R}$$

So, the Law of Sines can be applied to obtuse triangles.

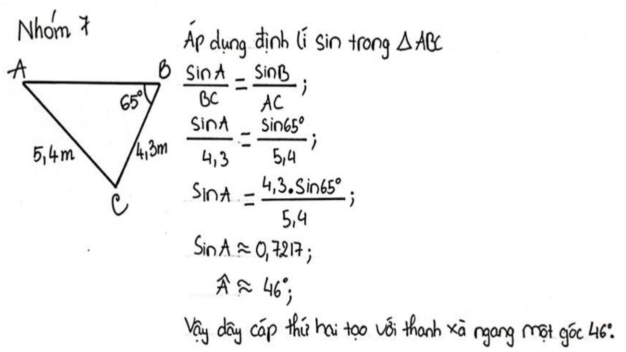
Figure 9. Group 6's presentation for the case where triangle ABC is an obtuse triangle

In the case of obtuse triangle ABC, students used the following properties to deduce the Law of Sines: Two opposite angles of a quadrilateral inscribed in a circle are supplementary, $sin(180° - \alpha) = sin\alpha$, proportionality property, transitivity property, inscribed angle intercepting the semicircle is a right angle. From there, it can be concluded that the Law of Sines applies not only to acute triangles but also to right triangles and obtuse triangles. Thus, in this activity, students performed vertical mathematization through the use of mathematical rules to build a general model, the Law of Sines, for all triangles.

Activity 4. Solve the Initial Problem

Teacher: Now we will return to solving the original real - life problem using the Law of Sines. Which group can volunteer to give the solution on the board?

The groups actively discussed for several minutes, and the results showed that all groups were able to solve the original real - life problem. Group 7 volunteers to to present their solution, as shown in Figure 10.



Group 7: Applying the Law of Sines in triangle *ABC*:
$\frac{sinA}{BC} = \frac{sinB}{AC}$; $\frac{sinA}{4.3} = \frac{sin65°}{5.4}$; $sinA = \frac{4.3 \times sin65°}{5.4}$;
$sinA \approx 0.7127$;
$\hat{A} = 46°$;
So the second cable makes a 46° degree angle with the crossbar.

Figure 10. Group 7's poster solution

The teacher asked group 7 a question: "Are the results of the problem reasonable compared to reality? Why?" Furthermore, Group 7 discusses with each other and answers: "The results of the problem are completely reasonable compared to reality because in any triangle, the angle opposite the side with the longer length has the larger measure. In this case, angle $\hat{A}$ is opposite side BC, and $BC < AC$, so the measure of $\hat{A}$ is smaller than the measure of $\hat{B}$, which is reasonable". The remaining groups agree with the solution and answer of group 7. The teacher concludes that group 7's presentation was completely reasonable.

In this activity 4, students used the "model - for" the Law of Sines ($sinA/BC = sinB/AC = sinC/AB = 1/2R$), and properties of proportions to solve the mathematical model (triangle ABC) of the original problem. Then, they consider and argue the reasonableness of the solution to the problem (the measure of angle A).

Activity 5. Apply the Law of Sines

The teacher distributes worksheet number 2 to students, which includes real - life problems and guiding questions.

Practical problem:

From the window of his room in downtown Saigon, Trung saw the monument of Tran Hung Dao, a brilliant military strategist who defeated Mongol invaders in the 13th century and became a cultural hero of modern Vietnamese people, as shown in Figure 11. The angle created by Trung's view to the top and to the foot of the Tran Hung Dao monument is $7°$. The angle created between the horizontal and Trung's line of sight to the top of the monument is $20°$. The horizontal distance between the Trung and the front of the statue is 71 m.

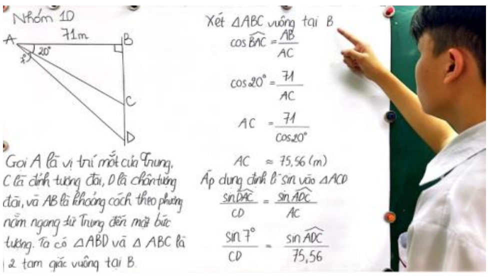
1. Construct a model to represent this problem.
2. Determine the height of Tran Hung Dao monument.
3. Determine the distance from Trung's eyes to the foot of the monument. (Round results to the nearest hundredths).

The system of guiding questions includes:

1. What is the problem that needs to be solved here?
2. What information is needed to solve the problem?
3. Construct a diagram describing the data of the problem.
4. Solving real - life problem initially boils down to solving which familiar problem?
5. Are the results of the problem reasonable in real – life? Why?

Figure 11. Monument of General Tran Hung Dao (Author's photo)

Upon receiving the study sheets, each group member independently researched and attempted to solve the assigned problem, guided by the teacher's system of directed questions. Following this individual effort, group members engaged in discussions to exchange ideas and collaboratively refine their solutions. The results indicated that all groups successfully constructed a "model of" the problem. However, during the application of the Law of Sines to solve the constructed model, only Group 10 encountered difficulties and was unable to arrive at a correct solution. To facilitate knowledge sharing and enhance collective understanding, the teacher invited Group 8 to present their solution on the board. This presentation provided an opportunity for peer learning and the reinforcement of key concepts through collaborative discussion.



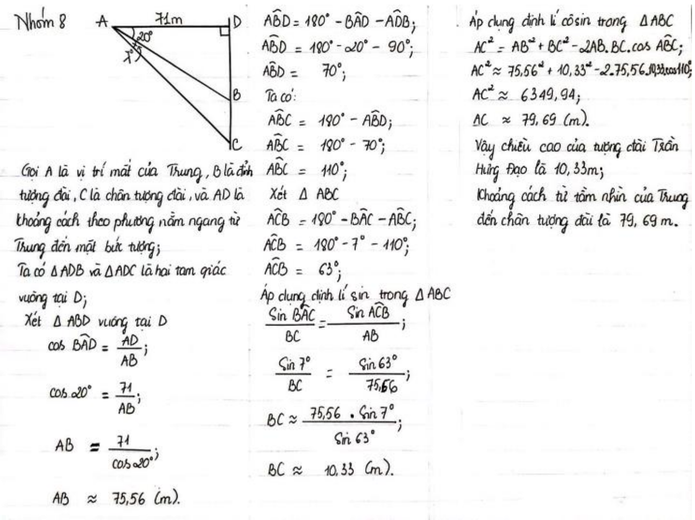
Group 10
Let *A* be Trung's eye position, *C* be the top of the statue, *D* be the foot of the statue, and *AB* be the horizontal distance from Trung to the face of the statue. The two angles *ABD* and *ABC* are two right triangles at *B*.
Consider triangle *ABC* right at *B*: $cos\widehat{BAC} = AB/AC$; $cos20° = 71/AC$; $AC = 71/cos20°$; AC ≈ 75.56 (m).
Applying the Law of Sines to triangle *ACD*: $\frac{sin\widehat{DAC}}{CD} = \frac{sin\widehat{ADC}}{AC}$; $\frac{sin7°}{CD} = \frac{sin\widehat{ADC}}{75.56}$.

Figure 12. Solution on poster of Group 10

In the solution of Group 10 (see Figure 12), the group in turn names $A, D,$ and C for Trung's eye position, the base of the monument, and the top of the statue. The projection of *A* on ray DC is B. According to the data of the problem, the group assigns the values $\widehat{CAD} = 7°$, $\widehat{CAB} = 20°$, $AB = 71$ m.

The teacher comments: "Group 10 uses the trigonometric ratio $cos\widehat{BAC} = AB/AC$ in triangle ABC right-angled at B to calculate the length of AC. However, when applying the Law of Sines to triangle ACD to calculate the height of the monument CD, you are stuck because you cannot calculate the measure of the angle $\widehat{ADC}$. Therefore, Group 10 cannot answer question 2 of the problem.

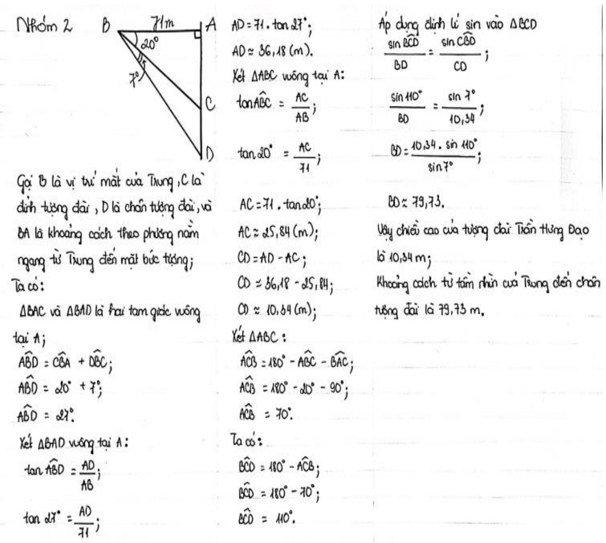
Then, the teacher calls Group 8 that successfully solved the problem to present their solution on the board, as shown in Figure 13. After attaching their group's poster to the board, the representative of Group 8 points to the solution, explaining how to do it and answering the teacher's guiding questions and the three questions of the problem. The other groups follow Group 8's explanation and completely agree with the solution. The teacher highly appreciate the solution of Group 8.



Group8: Let *A* be Trung's eye position, *B* be the top of the statue, *C* be the foot of the statue, and *AD* be the horizontal distance from Trung to the face of the statue. The two angles *ADB* and *ADC* are two right triangles at *D*;
Consider triangle *ABD* right at *D*
$cos\widehat{BAD} = AD/AB$; …; $\widehat{ACB} = 63°$;
Applying the Law of Sines to triangle *ABC*:
$\frac{sin\widehat{BAC}}{BC} = \frac{sin\widehat{ACB}}{AB}$; … ; $BC \approx 10.33$ (m).
Applying the Law of Cosine to triangle ABC:
$AC^2 = AB^2 + BC^2 - 2.AB.BC.cos\widehat{ABC}$; …;
$BC \approx 79.69$ (m).
the height of Tran Hung Dao monument is 10.33 m;
The distance from Trung's eyes to the foot of the monument is 79.69 m.

Figure 13. Solution on poster of Group 8

Next, the teacher calls Group 2 to present their solutions on the board (see Figure 14).



Group8: Let *B* be Trung's eye position, *C* be the top of the statue, *D* be the foot of the statue, and *AB* be the horizontal distance from Trung to the face of the statue.
ΔBAC and ΔBAD are two right triangles at *A*;
$\widehat{ABD} = \widehat{CBA} + \widehat{DBC}$; …; $\widehat{ABD} = 27°$;
Consider ΔBAD right at *A*: $tan\widehat{BAD} = AD/AB$; …; $AD \approx 36.18$ (m).
Consider ΔABC right at *A*: $tan\widehat{ABC} = AC/AB$; …; $CD \approx 10.34$ (m);
Consider ΔABC: $\widehat{ACB} = 180^0 - \widehat{ABC} - \widehat{BAC}$; …; $\widehat{ACB} = 70^0$.
We have: $\widehat{BCD} = 180^0 - \widehat{ACB}$; …; $\widehat{BCD} = 110°$.
Applying the Law of Sines to ΔBCD:
$\frac{sin\widehat{BCD}}{BD} = \frac{sin\widehat{CBD}}{CD}$; …; $BD = \frac{10.34 \times sin110°}{sin7^0}$;
$BC \approx 79.73$ (m).
The height of Tran Hung Dao monument is 10.34 m;
The distance from Trung's eyes to the foot of the monument is 79.73 m.

Figure 14. Solution on Group 2's poster

The teacher comments: "Group 2 has a different way of solving problems than Group 8. First, the group uses the trigonometric ratio $tang\ \widehat{ABD} = AD/AB$ in triangle ABD to calculate AD, then uses the $tang\widehat{ABC} = AC/AB$ to calculate AC, and from there calculate the length of CD, is the height of the monument. Then, the group calculates the measure of the angle $\widehat{ACB}$, and then deduces the measure of the angle $\widehat{BCD}$. Finally, the group uses the Law of Sines for triangle ACD to calculate the length of BD, which is the distance from Trung's eye to the foot of the monument.

The other groups focus on reading Group 2's solution while listening to the teacher's comments, and all agree with Group 2's solution. Next, the teacher notes that the difference in approximate values of the results in all groups' work is due to the greater or lesser number of approximation steps. However, it is necessary to agree on the approximate values $BC \approx 10.3$ and $AC \approx 79.7$. Finally, the teacher poses a question to the groups: "Are the monument's height of 10.3 m and the distance of 79.7 m from Trung's eye to the base of the monument consistent with reality?" Explain?".

Students argue that the results $BC \approx 10.3$ and $AC \approx 79.7$ are reasonable because the height of the monument is in fact almost as tall as a two-floor building and the distance from the monument to surrounding hotels is about 50 m. On the other hand, if we consider triangle ABC in the solution of group 8, we have: $79.7 - 75.6 < 10.33 < 79.7 + 75.6$. This satisfies triangle inequality theorem: $|AB - AC| < BC < AB + AC$.

In this activity 5, all groups successfully perform horizontal mathematization when they build a "model - of" the real - life problem, which are two triangles ADB and ADC right-angled at D (case of Group 8), and express on that model the measures of angles $\widehat{CAB}$, $\widehat{BAD}$ and the length of side AD. However, in the vertical mathematization, only Group 10 has difficulty determining the measure of the angle $\widehat{ACB}$, the remaining groups all successfully solve the problem using the model – for, the Law of Sines.

Discussion

In the process of conducting teaching experiments, students are tasked with addressing three key problems: (1) an initial real-life problem requiring solution-oriented thinking, (2) a general problem that facilitates the rediscovery of the Law of Sines, and (3) a real-life application problem where students apply the newly learned Law of Sines. To address these challenges, a mathematization process comprising four steps is integrated into five instructional activities focused on teaching the Law of Sines.

This teaching approach does not rely on historical phenomenology due to the difficulties in pinpointing the historical origins of the Law of Sines. Instead, it adopts a pure phenomenological framework, beginning with a contextualized scenario: calculating the angle between the crossbar and the cable of a disco ball. In the first step, students construct a triangle representing the beam and two cables of the disco ball, attempting to solve the problem using previously learned mathematical concepts, such as trigonometric ratios in right triangles. At this stage, the "model of" the Law of Sines has yet to emerge. To address this limitation, the subsequent step introduces a second context based on a triangular representation of the initial problem, accompanied by a system of guiding questions designed to help students discover the Law of Sines independently.

In Activity 1, the teacher gives an initial real-life problem and asks students to read, understand the problem, and determine what the problem requires. After that, students select the necessary data and construct a triangle ABC model for the problem, then they create a diagram including the steps to solve the problem. In this activity, the "model - of" the original problem is triangle ABC with altitude CH, and two relations $sin\hat{A} = CH/AC$ and $sin\hat{B} = CH/BC$. In this activity, horizontal mathematization is demonstrated through reading and understanding problems, converting the data of the problem into a triangle ABC model and suggesting solutions. Therefore, the horizontal mathematicization operation and steps 1, 2 and 3 of the mathematicization process are revealed.

In Activity 2, from the triangle model of the initial problem, students develop it into a general problem of finding the angle of a triangle when knowing the length of the opposite side and the measure of any remaining angle and the length of the side opposite that angle. Through solving general problem, by eliminating the CH quantity in the trigonometric ratio relations, students find the relation $sin\hat{A}/BC = sin\hat{B}/AC$ which is part of the "model - for" the Law of Sines. From there, by through subsequent questions, the teacher guides students to reinvent the complete "model - for" the Law of Sines for an arbitrary triangle. Therefore, vertical mathematization and step 3 of the mathematization process are revealed.

In Activity 3, the "model-for" is formalized as the Law of Sines. Students then extend their understanding by proving that the Law of Sines holds true for both right and obtuse triangles. This process unveils the concept of vertical mathematization and corresponds to Step 3 of the mathematization process. Furthermore, in Activity 4, students revisit the initial problem, now equipped with the "model-for" represented by the Law of Sines. They solve the problem again and evaluate whether the results align with real-world expectations. At this stage, vertical mathematization is further reinforced, and Steps 3 and 4 of the mathematization process are demonstrated.

Finally, in Activity 5, the teacher presents a new real-life problem that requires students to engage in multiple stages of mathematical reasoning. Students are tasked with reading and understanding the problem, identifying its requirements, selecting relevant data, and constructing a "model-of" the situation. This step highlights horizontal mathematization. Subsequently, students apply the "model-for" (the Law of Sines) alongside other mathematical tools to solve the constructed model. This process reveals vertical mathematization. After obtaining a solution, students critically assess whether the results are reasonable within the context of the real-world scenario. By the conclusion of this activity, all four steps of the mathematization process are explicitly demonstrated.

CONCLUSION

This study explored the application of the RME approach in teaching the Law of Sines, focusing on students' engagement in the guided reinvention process. The experimental results indicate that students actively participated in the mathematization process, converting real-life problems into mathematical models through horizontal mathematization and solving these models using vertical mathematization. This process enabled students to understand the significance and practical utility of the Law of Sines while appreciating mathematics as a human activity with strong connections to real-life applications. Moreover, the findings demonstrate that a well-designed teaching process based on RME principles, including didactical phenomenology, emergent model heuristics, and guided reinvention, facilitates students' ability to rediscover mathematical concepts independently.

Despite these positive outcomes, the study identifies several limitations. A notable limitation is the observed discrepancy in student performance between vertical and horizontal mathematization, with vertical mathematization yielding better results. This disparity can be attributed to the Vietnamese educational context, which prioritizes problem-solving and knowledge application over rediscovery and contextual understanding. Furthermore, the cyclical and branching nature of mathematization in applying the Law of Sines suggests that the interplay between horizontal and vertical processes requires further exploration. The study also highlights the need for examining the influence of students' prior knowledge on their ability to reinvent new concepts, as well as the critical role of teachers in designing effective activities and guiding questions that align with RME principles.

To expand upon this research, future studies should focus on designing and implementing RME-based teaching strategies for high school mathematics, as most current research has been limited to elementary and middle school levels. Incorporating historical epistemology into the design of mathematical problems could enhance the authenticity of the learning experience, allowing students to better understand the origins and development of mathematical concepts like the Law of Sines.

Additionally, conducting experiments in diverse educational contexts with students of varying mathematical abilities could increase the generalizability and applicability of the findings. By addressing these areas, future research can provide deeper insights into optimizing the teaching and learning of mathematical topics through the RME framework.

Declarations

Author Contribution	:	QAN: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Project administration, Validation, Visualization, Writing - original draft, and Writing - review & editing. NGN: Methodology, Project administration, Visualization, Supervision, Writing – review & editing.
Funding Statement	:	Securing funding.
Conflict of Interest	:	No conflict of interest.

REFERENCES

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects - State, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, *22*(1), 37–68. https://doi.org/10.1007/BF00302716

Boaler, J. (1993). Encouraging the transfer of ‘school'' mathematics to the “real world” through the integration of process and content, context and culture.’ *Educational Studies in Mathematics*, *25*(4), 341–373. https://doi.org/10.1007/BF01273906

Da, N. T. (2022). Designing a teaching model based on the Realistic Mathematics Education (RME) approach and its application in teaching calculus. *Journal of Mathematics and Science Teacher*, *2*(1), em006. https://doi.org/10.29333/mathsciteacher/11918

Danh Nam, N. (2020). Some problems in realistic mathematics education. *Vietnam Journal of Education*, *487*(1), 15–21. https://mysite.tnu.edu.vn/files/users/nguyendanhnam/487_Nguyn_Danh_Nam.pdf

De Lange, J. (2006). Mathematical literacy for living from OECD-PISA perspective. Tsukuba Journal of Educational Study in Mathematics. *Special Issue on The APEC- TSUKUBA International Conference “Innovative Teaching Mathematics through Lesson Study,” 25*, 13–35. http://www.criced.tsukuba.ac.jp/math/apec2006/

Felton, M. D. (2010). News & amp; Views: Is math politically neutral? *Teaching Children Mathematics*, *17*(2), 60–63. https://doi.org/10.5951/TCM.17.2.0060

Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics*, *1*(1–2), 3–8. https://doi.org/10.1007/BF00426224

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*(3–4), 413–435. https://doi.org/10.1007/BF00302305

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Reidel Publishing Company.

Freudenthal, H. (2002). *Revisiting Mathematics Education: China lectures* (Vol. 9). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47202-3

Gravemeijer, K. (1994a). *Developing realistic mathematics education*. CD β Press. https://www.fisme.science.uu.nl/publicaties/literatuur/1994_gravemeijer_dissertation_0_222.pdf

Gravemeijer, K. (1994b). *Modelling two-digit addition and subtraction with an empty number line. In: T. Breiteig, G. Kaiser-Messmer and I. Huntley. Mathematics works – Mathematical Modelling in the Classroom*. Ellis Horwood / Simon and Schuster.

Gravemeijer, K. (1999). How Emergent Models May Foster the Constitution of Formal Mathematics. *Mathematical Thinking and Learning*, *1*(2), 155–177. https://doi.org/10.1207/s15327833mtl0102_4

Gravemeijer, K., & Doorman, M. (1999). Context Problems in Realistic Mathematics Education: A Calculus Course as an Example. *Educational Studies in Mathematics*, *39*(1/3), 111–129. https://doi.org/10.1023/A:1003749919816

Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: A mathematician on didactics and curriculum theory. *Journal of Curriculum Studies*, *32*(6), 777–796. https://doi.org/10.1080/00220270050167170

Hendroanto, A., van Galen, F., Van Eerde, D., Prahmana, R. C. I., Setyawan, F., & Istiandaru, A. (2018). Photography activities for developing students' spatial orientation and spatial visualization. *Journal of Physics: Conference Series, 943*(1), 012029. http://dx.doi.org/10.1088/1742-6596/943/1/012029

Inci, A. M., Peker, B., & Kucukgencay, N. (2023). *Realistic mathematics education. In book: Current Studies in Educational Disciplines 2023 (pp.66-83) Edition 1, Chapter 5. Publisher: ISRES Publishing.* ISRES Publishing. https://www.researchgate.net/publication/377209600_Realistic_Mathematics_Education

Johnson, P., Almuna, F., & Silva, M. (2022). The role of problem context familiarity in modelling first-order ordinary differential equations. *Journal on Mathematics Education*, *13*(2), 323–336. https://doi.org/10.22342/jme.v13i2.pp323-336

Jupri, A., & Drijvers, P. (2016). Student Difficulties in Mathematizing Word Problems in Algebra. *EURASIA Journal of Mathematics, Science and Technology Education*, *12*(9), 2481–2502. https://doi.org/10.12973/eurasia.2016.1299a

Kant, D., & Sarikaya, D. (2021). Mathematizing as a virtuous practice: different narratives and their consequences for mathematics education and society. *Synthese*, *199*(1–2), 3405–3429. https://doi.org/10.1007/s11229-020-02939-y

Larsen, S. (2018). Didactical phenomenology: the engine that drives realistic mathematics education. *For the Learning of Mathematics*, *38*(3), 25–29. https://www.jstor.org/stable/26548508

Le, T. T., Pham, A. G., & Nguyen, T. T. (2021). Applying realistic mathematics education theory in teaching: some challenges, principles and recommendations. *Vietnam Journal of Education*, *494*(2), 37–43. https://tcgd.tapchigiaoduc.edu.vn/index.php/tapchi/article/view/22

Loc, N. P., Tran, N., & Tien, T. (2020). Approach To Realistic Mathematics Education In Teaching Mathematics: A Case Of Cosine Theorem-Geometry 10. *International Journal of Scientific & Technology Research*, *8*, 1173–1178. www.ijstr.org

Nguyen, T. T., Trinh, T. P. T., Ngo, H. T. V., Hoang, N. A., Tran, T., Pham, H. H., & Bui, V. N. (2020). Realistic mathematics education in Vietnam: Recent policies and practices. *International Journal of Education and Practice*, *8*(1), 57–71. https://doi.org/10.18488/journal.61.2020.81.57.71

Nkambule, T. (2009). Teaching and learning linear programming in a grade 11 multilingual mathematics class of English language learners: exploring the deliberate use of learners' home language. In *Pytagoras* (Vol. 71). https://wiredspace.wits.ac.za/server/api/core/bitstreams/2fa4b36b-2db3-4630-b535-41700db0041e/content

Pierce, R. U., & Stacey, K. C. (2006). Enhancing the image of mathematics by association with simple pleasures from real world contexts. *ZDM*, *38*(3), 214–225. https://doi.org/10.1007/BF02652806

Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (2005). Advancing Mathematical Activity: A Practice-Oriented View of Advanced Mathematical Thinking. *Mathematical Thinking and Learning*, *7*(1), 51–73. https://doi.org/10.1207/s15327833mtl0701_4

Risdiyanti, I., & Prahmana, R. C. I. (2020). The learning trajectory of number pattern learning using "Barathayudha" war stories and uno stacko. *Journal on Mathematics Education, 11*(1), 157-166. http://doi.org/10.22342/jme.11.1.10225.157-166

Streefland, L. (1985). Wiskunde als activiteit en de realiteit als bron (Mathematics as an activity and reality as a source). *Nieuwe Wiskrant*, *5*(1), 60–67. http://www.fisme.science.uu.nl/wiskrant/artikelen/artikelen00-10/051/0501september_streefland.pdf

Streefland, L. (1991). *Fraction in Realistic Mathematics Education, A paradigm of developmental research*. Kluwer Academic Publisher.

Treffers, A. (1987). *Three dimensions. A model of goal and theory description in mathematics instruction: The Wiskobas Project*. Springer Dordrecht.

Treffers, A. (1991). Didactical background of a mathematics program for primary education. In: L. Streefland (ed.). Realistic mathematics education in primary school. In *Realistic mathematics education in primary school* (pp. 21–56). CDß press/Freudenthal Institute, Utrecht University, Utrecht, The Netherlands. https://www.fisme.science.uu.nl/publicaties/literatuur/1991_streefland_0-209.pdf

Van Den Heuvel-Panhuizen, M. (2000). *Mathematics education in the Netherlands: A guided tour 1. Freudenthal Institute CD-rom for International Commission on Mathematics Education (ICME) 9*. https://p4mriunismuh.wordpress.com/wp-content/uploads/2010/08/mathematics-education-in-the-netherlands.pdf

Van Den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, *54*(1), 9–35. https://doi.org/10.1023/B:EDUC.0000005212.03219.dc

Van Den Heuvel-Panhuizen, M. (2005). The role of contexts in assessment problems in mathematics. *For the Learning of Mathematics*, *25*(2), 2–23.

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2014). Realistic Mathematics Education. In *Encyclopedia of Mathematics Education* (pp. 521–525). Springer Netherlands. https://doi.org/10.1007/978-94-007-4978-8_170

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic Mathematics Education. In *Encyclopedia of Mathematics Education* (pp. 713–717). Springer International Publishing. https://doi.org/10.1007/978-3-030-15789-0_170

Van den Heuvel-Panhuizen, M., & Wijers, M. (2005). Mathematics standards and curricula in the Netherlands. *Zentralblatt Für Didaktik Der Mathematik*, *37*(4), 287–307. https://doi.org/10.1007/BF02655816

Vietnam Mathematics General Education Curriculum, Pub. L. No. 32/2018/TT-BGDĐT (2018).

Yackel, E., Stephan, M., Rasmussen, C., & Underwood, D. (2003). Didactising: Continuing the work of Leen Streefland. *Educational Studies in Mathematics*, *54*(1), 101–126. https://doi.org/10.1023/B:EDUC.0000005213.85018.34