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Learning obstacles and the didactical design for teaching the limit of function in a Calculus course

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Abstract

Students often face difficulties in understanding the concept of limits in functions, a challenge that arises due to the abstract nature and complexity of the topic. Despite being familiar with the procedural steps, students may fail to grasp the underlying meaning of limits. This gap in comprehension leads to significant learning obstacles. As such, there is a critical need for effective didactical designs that can enhance the teaching and learning of this concept. This study aims to address this issue through a Didactical Design Research (DDR) approach, which is structured into three phases. The first phase involves a preliminary didactical design analysis, followed by administering a diagnostic test on the limit of functions to 26 third-semester students (Group 1) who have already completed a differential calculus course. This diagnostic test helps identify the initial learning obstacles. In the second phase, a didactical design is developed to address these obstacles, and it is then implemented with 33 first-semester students (Group 2) enrolled in a Differential Calculus course to evaluate the impact of the design. Data analysis is conducted based on the scores from the written diagnostic test, categorizing them into three levels of ability. The findings reveal that the primary learning obstacle for students is the formal definition of limits, and the identified obstacles are epistemological, psychological ontogenic, instrumental ontogenic, and conceptual ontogenic. The results of implementing the didactical design demonstrate a significant improvement in students' understanding of limits, as evidenced by a reduction in the learning obstacles encountered. This research contributes to the development of more effective didactical approaches for teaching complex mathematical concepts, offering a potential model for addressing similar learning challenges in other abstract topics.

Keywords: Didactical Design Research, Differential Calculus, Learning Obstacle, Limit of Function

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Calculus holds a significant position in both secondary and higher education (Rasmussen et al., 2014). At the secondary education level, it is a compulsory subject, with concerns being raised regarding the emphasis on rote memorization at the expense of conceptual understanding (Grossfield, 2020). In higher education, calculus is a prerequisite course in various disciplines such as engineering, mathematics, and the natural sciences, typically offered in the first semester. Furthermore, individuals pursuing careers in education, engineering, medicine, economics, science, and mathematics endeavor to master and understand the fundamental principles and methodologies of calculus. The "ICME-13 Topical Survey," focusing primarily on topics related to limits, derivatives, and integrals, provides a global perspective on significant advancements in the field of calculus education and pedagogy (Bressoud et al., 2016).

Limits, a fundamental topic in calculus, significantly impact the understanding of subsequent concepts such as derivatives and integrals (Buyukköroglu, 2006). The limits of functions at a specific point and the limits of infinite functions are essential concepts in both differential and integral calculus, which are extensively applied. A comprehensive understanding of the limit of a function necessitates familiarity with absolute value as a measure of distance on the number line, inequalities as a measure of proximity, and the various properties of real functions as mathematical objects (Martono, 1999). Consequently, the real number system, inequality theory, absolute values, functions, and graphs serve as foundational or precalculus topics before delving into the concept of limits (Varberg et al., 2013).

A widely used calculus textbook in Indonesian universities is authored by Varberg et al. (Purnomo et al., 2017; Supriadi et al., 2020), which introduces the material on limits by first providing an intuitive or informal understanding of the concept, followed by a formal definition of limits. Research by Gürbüz et al. (2018) reveals that only a small percentage of individuals grasp the formal definition of limits effectively through traditional instruction. The inherent complexity of the limit definition has led many students to develop informal understandings of the concept, prompting the need for a formal definition to guide them. Adiredja (2021) suggests that some reasoning patterns rely on an intuitive approach, where students select values of x close to a to determine the limit. Similarly, Boester (2008) developed narratives to tap into students' intuitive knowledge, helping them connect prior knowledge with a more formal comprehension of limits. Oehrtman (2008) noted that the majority of introductory calculus courses and textbooks begin with an intuitive approach to teaching calculus. This intuitive approach to limits often involves expressions such as "x approaches a, f(x) approaches L." While formal definitions are introduced to demonstrate essential properties of limits, these definitions are neither emphasized nor consistently integrated when progressing to subsequent topics, including those defined in terms of limits. For example, the definition of a derivative is rarely presented using the epsilon-delta formalism in introductory calculus courses. The overarching goal is to provide an accessible and broad introduction to key concepts throughout the calculus curriculum.

Nagle et al. (2017) stated that the main role of limit concepts in understanding introductory calculus, and previous findings highlight students' difficulties and misunderstandings regarding limits that are troubling. Researchers consistently describe students' understanding of limits as procedural, defining the concept of limits by the procedures used to calculate limit values. Nagle et al. contend that students may define a limit as a number calculated by one of the various procedures performed on a function (e.g. 'division technique' or 'direct substitution'). If $f(x) = \lim_{x \to 3} \frac{x^2-5x+6}{x-3}$, some students can use this procedural knowledge to determine $\lim_{x \to 3} \frac{x^2-5x+6}{x-3} = 1$, but fail to apply this knowledge to estimate its value from $f(2.99)$. Research also shows that some students fail to conceptualize the limit as a number, viewing it as a process by which the function value approaches the real number L, rather than as a product, specifically the number L which is the result of this process.

The concept of limits in mathematics is abstract and presents significant challenges for students, requiring advanced cognitive abilities (Juter, 2007; Bansilal & Mkhwanazi, 2021). Several studies on limits highlight that students continue to struggle with solving limit-related problems. Denbel (2014) concluded that students' knowledge and understanding are often fragmented, relying heavily on isolated facts, routine calculations, and the memorization of algorithms and procedures. As a result, students' conceptual understanding of limits, continuity, and infinity remains insufficient. Jordaan (2005) identified a common misconception in which students equate the limit value with the function value at a point, assuming that limits can be determined by the substitution method. Additionally, Hong and Choi (2019) conducted studies on students' comprehension of the limit concept, revealing that many students face learning challenges when engaging with limits. These challenges include the mistaken belief that only continuous functions have limits, the misconception that limits are approximations and may not be equal to a number, and difficulties in grasping the concept of infinite processes. The reliance on intuition and the application of procedures without full comprehension often contribute to students' restricted understanding and misconceptions (Swinyard, 2011; Berry & Nyman, 2003). Tall and Vinner (1981) argued that students' cognitive structures, visual images, and properties—collectively known as concept images—often hinder their ability to understand formal definitions of concepts. Despite these difficulties, students may still solve calculus problems by following procedures, even when they lack a comprehensive understanding of the underlying concepts. This approach can be a significant source of learning obstacles.

Learning obstacles refer to conditions that impede students' ability to acquire new knowledge during the learning process, often leading to learning difficulties. These obstacles can be identified through students' errors (Suryadi, 2019). Ontogenic obstacles relate to students' mental preparedness and cognitive maturity, which influence their ability to absorb knowledge. These obstacles are categorized into three types: psychological ontogenic obstacles, instrumental ontogenic obstacles, and conceptual ontogenic obstacles. Psychological ontogenic obstacles are associated with motivation and interest in the subject matter, instrumental ontogenic obstacles involve the technical aspects of problem-solving, and conceptual ontogenic obstacles arise when a concept's level in the learning process does not align with the students' actual learning experiences (Lutfi et al., 2021).

The second type of learning obstacle is the didactical obstacle, which pertains to the sequence of learning activities undertaken by students or the manner in which the lecturer presents the material. Didactical obstacles become evident when the actual learning process diverges from the content anticipated in the theoretical framework. These obstacles can be identified through the structure and progression of material delivery. Additionally, didactical obstacles may arise from the teaching materials prepared and used by the instructor during the learning process. The epistemological obstacle is related to the tasks assigned to students, focusing not only on their ability to apply procedural knowledge but also on their capacity to construct understanding and ways of thinking. These obstacles may result from incomplete or insufficient information acquired by students, which can hinder their ability to solve problems when faced with new or unfamiliar situations, such as when questions are presented without varying difficulty levels.

Analyzing learning obstacles in mathematics education is essential for improving teaching and learning outcomes. Common errors in learning inverse functions include conceptual, procedural, calculation, and conclusion errors, which are influenced by both internal and external factors (Perbowo & Anjarwati, 2017). Prospective mathematics teachers encounter ontogenic, didactic, and epistemological obstacles when designing lesson plans, underscoring the need for improved teaching strategies and lesson designs (Prabowo et al., 2022). These studies highlight the importance of identifying and addressing learning obstacles to enhance mathematics education.

However, previous research has primarily overlooked specific learning obstacles in higher education and how these obstacles impact the learning process. This gap in the existing literature suggests a need for further exploration, particularly in the context of improving learning through instructional designs that address students' learning obstacles. By understanding the challenges students face, lecturers can focus on the limits of functions that are most likely to cause difficulties and develop strategies to overcome them. The implications of this research demonstrate that understanding learning obstacles can improve lecturers' awareness of university-level learning processes and assist in the development of targeted interventions, known as didactical designs, that foster more productive and engaging learning environments (Hendriyanto et al., 2024).

The didactical design is a learning plan based on the Theory of Didactical Situations (TDS) by Brousseau (2002), which consists of four stages: action, formulation, validation, and institutionalization. The teaching and learning process is structured around these stages. In the action stage, students are encouraged to develop new hypotheses. In the formulation stage, students engage in discussions, activities, and interactions with other groups to achieve the learning objectives. During the validation stage, the instructor, as a theorist, evaluates the theorems students have generated in the action and formulation phases. Institutionalization occurs when students are able to apply their newly acquired knowledge to solve problems and transform their existing knowledge into new insights, reinforced by the lecturer's validation of their understanding.

This research aims to address three key research questions: What are the most common learning obstacles faced by students in terms of content? How are students' learning obstacles related to the concept of limits of functions? And how can the didactical design for teaching limits of functions be implemented effectively? This study contributes to the identification of learning obstacles in the teaching of limits of functions and provides insights into which obstacles appear most frequently within this subtopic. The findings will enable lecturers to offer targeted instruction and interventions that can help alleviate the difficulties and obstacles students encounter in learning this concept.

METHODS

Research Design

This study employs a qualitative research approach within the framework of Didactic Design Research (DDR). DDR is grounded in its didactic nature as an art, science, and epistemology in the context of knowledge dissemination and acquisition, aiming to foster students' independence in producing new knowledge as justified true beliefs (Suryadi, 2019). DDR operates on two primary paradigms: the interpretive and the critical. The interpretive paradigm focuses on the diffusion and acquisition of knowledge, where researchers identify both existing and emerging learning barriers. In contrast, the critical paradigm seeks transformative change by proposing didactical designs that influence the teaching and learning process. The research in this study follows three distinct stages as outlined in Figure 1.

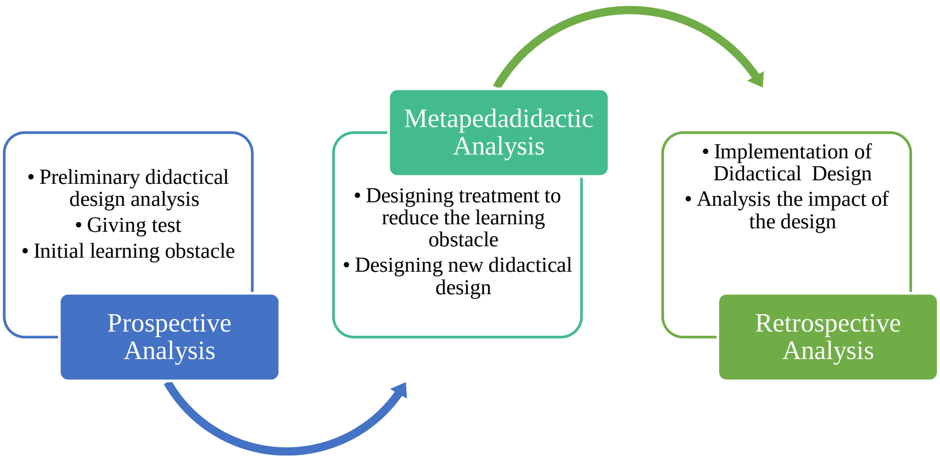


Figure 1. DDR stages (Suryadi, 2019)

Participants

The first phase of the research was conducted over a period of approximately two months, from November 1, 2023, to January 15, 2024, at Universitas Khairun, Ternate City, Indonesia. A purposive sampling technique was applied to select participants. Purposive sampling ensures the inclusion of specific cases that are relevant to the research question (Campbell et al., 2020). The selection criteria for participants are as follows:

1. Enrollment in regular Mathematics Education classes, Semester 1.
2. Prior completion of the limit of functions topic in the Differential Calculus course.
3. Willingness to attend a written test on November 6, 2023.

The initial group, comprising 26 students, undertook a written test on the limit of functions. Following the test, in-depth interviews were conducted with three students selected from the group, representing high, moderate, and low levels of proficiency. The selection of these three students was intended to explore their thought processes, responses, and identify potential learning obstacles related to the concept of limits. This phase provides valuable insights into the learning challenges within the Calculus classroom, informing future teaching strategies. The second phase of the study took place from October 4 to November 25, 2024, and involved 33 first-semester students enrolled in the Differential Calculus course for the implementation of the didactical design.

Data Collection and Analysis

A diagnostic test was utilized to identify learning obstacles related to the concept of limits. The test comprised four descriptive questions addressing key indicators of the limits concept. Participants were required to respond to essay questions demonstrating their understanding of function limits. Expert judgments were employed to validate the test instrument (Colson & Cooke, 2018). The test was reviewed and validated by two expert lecturers specializing in mathematical analysis and mathematics education. Their evaluation ensured the clarity and appropriateness of the test items.

Students' responses to the written test were evaluated using holistic scoring guidelines, divided into four distinct categories (Moskal, 2000). In addition to the diagnostic test, a didactical design was developed to address the learning obstacles identified from the diagnostic results. The validation of this didactical design was conducted through a focus group discussion (FGD) involving experts in DDR, algebra, mathematical analysis, and mathematics education.

The test indicators are based on content from three core textbooks: Calculus by Varberg, Purcell, and Rigdon (2013); Martono (1999); and Stewart (1999). Table 1 presents the indicators and scoring categories used for the assessment of the limit of functions.

Table 1. The indicators for each material on the limit of function

Contents	Indicator	Scoring Categories
Limit at a point	To find limit value using factoring	1. Finding answers correctly using factoring 2. Most of the answers reached the correct answer but there were still mistakes 3. Only a small number of answers reach the correct answer 4. No Answer
One-sided limits	To find left and right-hand limits	1. Finding the answer function value of left limits and right limits correctly

		2. Most of the answers reached the correct answer but there were still mistakes 3. Only a small number of answers reach the correct answer 4. No Answer
Formal Prove	To prove that a limit exists using ε-δ definition.	1. Finding the correct answer with a formal proof. 2. Most of the answers reached the correct answer but there were still mistakes 3. Only a small number of answers reach the correct answer 4. No Answer
Continuity	To determine whether a function is continuous or discontinuous.	1. Finding answer regarding the continuity function. 2. Most of the answers reached the correct answer but there were still mistakes 3. Only a small number of answers reach the correct answer 4. No Answer

The data analysis for the written test was conducted based on predefined scoring categories. Each participant's total score from the written test was categorized into three levels of proficiency. These categories for the limit of function test results were established using a holistic scoring rubric (Nurhayati et al., 2023). Following the categorization, one participant from each proficiency group was selected to represent their respective group for an in-depth interview.

After categorization, students were grouped into three proficiency levels. One participant from each group was chosen to participate in an in-depth interview, which aimed to explore their problem-solving experiences, with a particular focus on how the students approached and answered the test questions, as well as to identify any learning obstacles encountered (Sari et al., 2024). An overview of the entire research process is depicted in Figure 2.

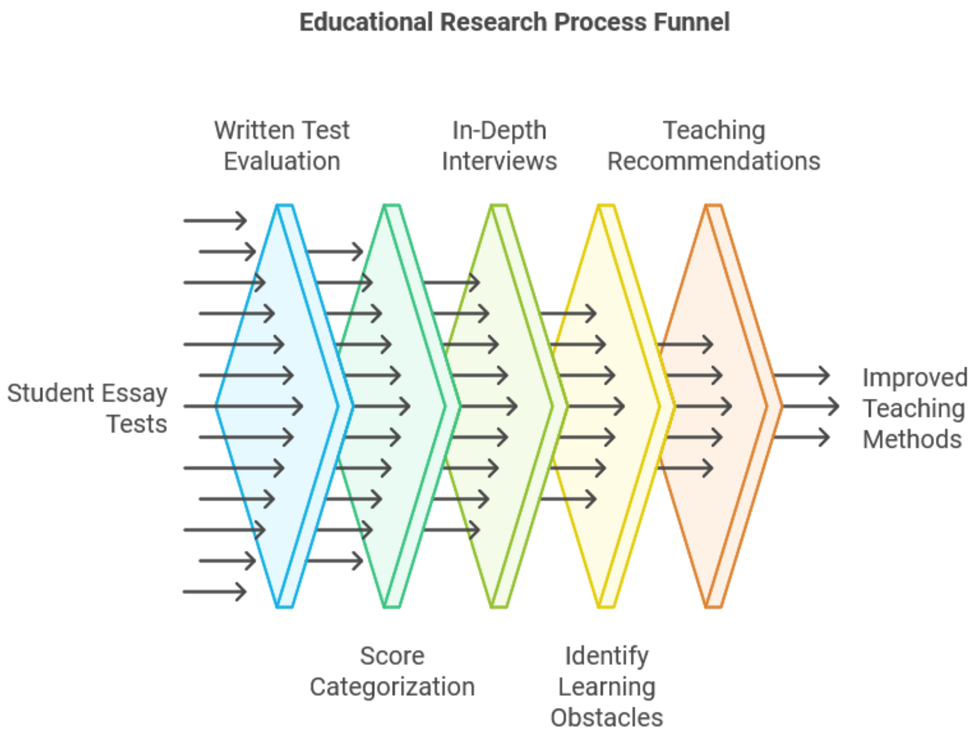


Figure 2. The research process at every level

RESULTS AND DISCUSSION

Participant Written Test Results

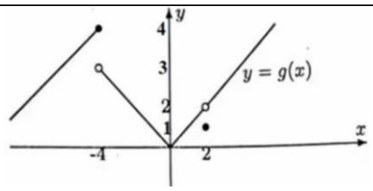
The results of the written tests administered to all participants regarding the limit at a point are presented in Table 2.

Table 2. Participants' written test on the limit at a point

Question	Answer Descriptions	Participants
Find $\lim_{x\to 3} \frac{x^2-x-6}{x-3}$	Correctly identifying the answer through factoring.	P10, P12, P13, P15, P17, P18, P19, P20, P24, P25, P26
	Most responses were correct, though some errors persisted.	P1
	A small number of responses were correct	P2, P3, P4, P5, P6, P7, P8, P9, P11, P14, P16, P21, P22, P23
	No answer provided	-

The results of the written tests regarding one-sided limits are shown in Table 3.

Table 3. Participants' written test on one-sided limits

Question	Answer Descriptions	Participants
	Correctly determining the left-hand and right-hand limits	P10, P12, P13, P18, P19, P23
For the function above, find the indicated limit or function value	Most responses were correct, though some errors persisted	P1, P6, P15, P20, P22
a. $\lim_{x\to 2(-)} g(x)$ b. $\lim_{x\to 0(-)} g(x)$ c. $\lim_{x\to -4(+)} g(x)$ d. $\lim_{x\to -4(-)} g(x)$ e. $g(2)$	Only a few responses were correct	P2, P3, P4, P5, P7, P8, P9, P11, P14, P16, P17, P21, P24, P25, P26
	No answer provided	-

The results of the written tests on the formal definition are displayed in Table 4.

Table 4. Participants' written test on the formal definition

Question	Answer Descriptions	Participants
Prove in a formal way that $= 10$	Finding the answer correctly with a formal definition of the limit of the function	P10, P12, P13, P18, P19
	Most of the answers reached the correct answer, but there were still mistakes.	P20, P22

Question	Answer Descriptions	Participants
	Only a small number of answers reached the correct answer.	P1, P2, P3, P4, P6, P8, P11, P15, P16, P21, P23, P24, P25, P26
	No Answer	P5, P7, P14, P17, P9

The results of the written tests on continuity are presented in Table 5.

Table 5. Participants' written test regarding continuity

Question	Answer Descriptions	Participants
f (t) = $\begin{cases} t-3, t \leq 3 \\ 3-t, t > 3 \end{cases}$ Investigate if f (t) is continuous or not. Tell the reason.	Correctly identifying the continuity of the function	P16, P12, P13, P15, P18, P19, P20, P22, P23, P24, P25, P26
	Most responses were correct, though some errors persisted	P1, P4, P8
	Only a few responses were correct	P2, P3, P5, P6, P7, P9, P11, P17
	No answer provided	P5, P14, P16

The data presented in Table 2 until 5 is further illustrated in Figure 3, which highlights the most significant learning obstacles encountered by the students based on the content assessed.

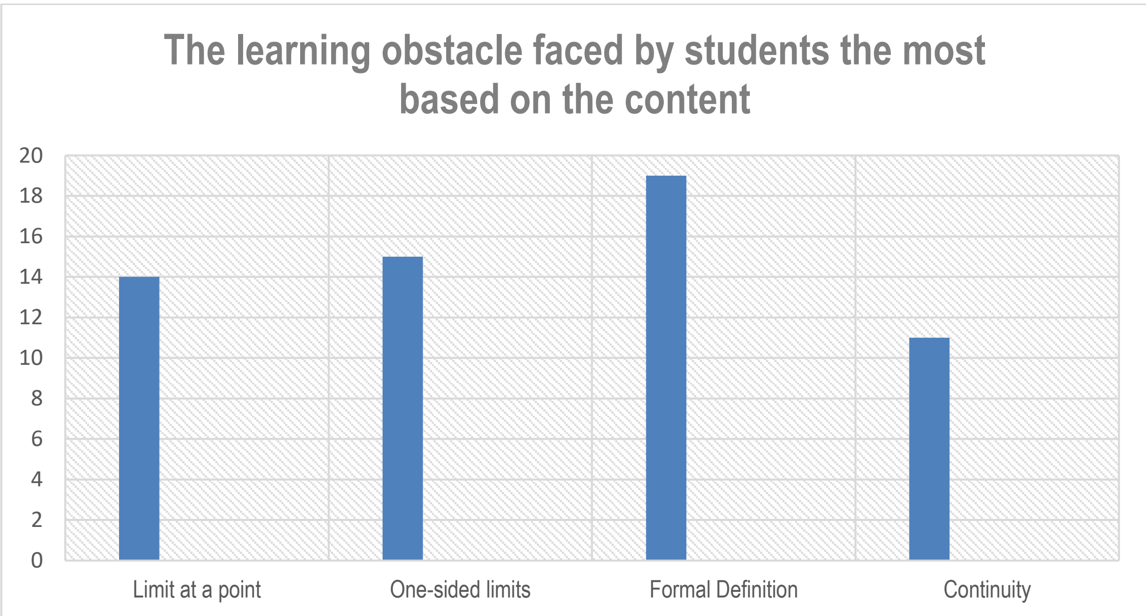


Figure 3. The most frequent learning obstacle faced by students in Group 1 based on content

Figure 3 illustrates that formal proofs of limits were the most frequent source of learning obstacles among students. This conclusion is derived from the number of participants who either provided incorrect responses or failed to answer the question altogether, as evidenced by two key response categories: a small number of correct answers and no answer provided. These findings suggest that instructors must dedicate additional focus to teaching the concept of formal limits in order to mitigate learning obstacles effectively.

The Test Results of Participants for Interview and Learning Obstacles

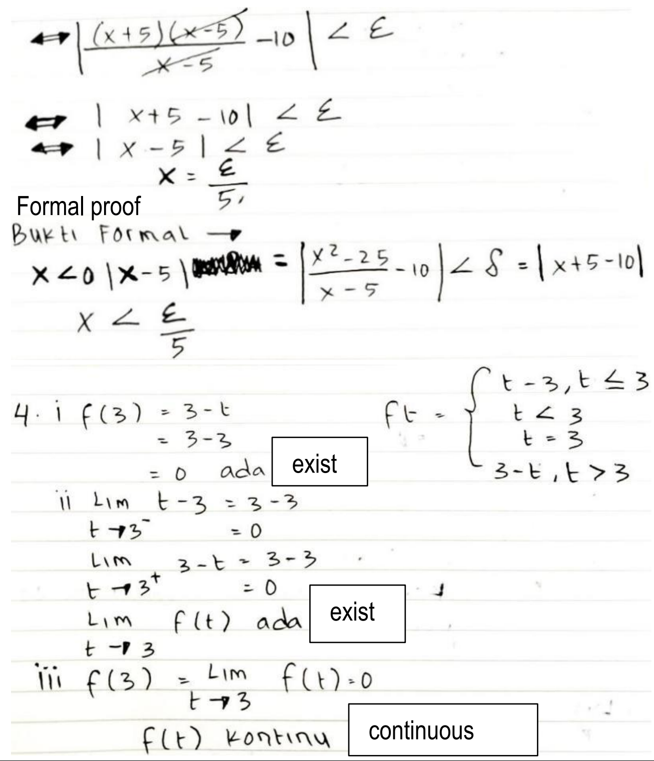
Three participants, representing each level of ability—high, moderate, and low—were selected for the interview analysis. These participants were labeled as P20 (high ability), P22 (moderate ability), and P17 (low ability). This section aims to address the second research question: How do the students describe their learning obstacles in the topic of limits of functions?

High-Ability Student: P20

Table 6 explains the interview results with P20. It was found that P20 did not experience any learning obstacles in relation to questions 1 and 4. However, in question 2(d), P20 encountered a misinterpretation. Although P20 understood that the values of -4^- and -4^+ are not the same due to the discontinuity of the function graph, P20 mistakenly assumed the question was asking about $f(x)$ rather than the limit. This led P20 to experience an epistemological obstacle, as they could not differentiate between the limit notation $\lim_{x\to\alpha} f(x)$ and the function notation $f(x)$.

Table 6. The interview results of high-ability student

P20 Answer	The Conclusion of Interview Results
1. $\lim_{x\to3}\frac{x^2-x-6}{x-3} = \lim_{x\to3}\frac{(x+2)(x-3)}{x-3}$ $= \lim_{x\to3} x+2$ $= 3+2$ $= 5$	P20 suggests solving limit problems through substitution. If the result is 0 or undefined then use factoring. P20 thinks that the way to solve limit problems is to use substitution. If the result is 0 or undefined, then use factoring.
2. a. $\lim_{x\to2^-} f(x) = 2$ b. $\lim_{x\to0^-} f(x) = 0$ c. $\lim_{x\to-4^+} f(x) = 3$ d. $\lim_{x\to-4^-} f(x) =$ Tidak ada karena memiliki 2 jawaban yg berbeda e. $f(2) = 1$ Does not exist, because it has two different answers.	The left limit of the point x = 2 is 2. Answer: 0 because 0 is neutral and the limit direction also points to 0. Because the graph is disconnected, the limit value from the right goes to 3 Because it is disconnected, the limit value does not exist. There are points 3 and 4, so the answer is none. The full point is one, so the answer is one. The answer is not an empty point.
3. $\lim_{x\to5}\frac{x^2-25}{x-5} = 10$ $x<0\ \lvert x-5\rvert<\delta \Rightarrow x<0\ \left\lvert\frac{x^2-25}{x-5}-10\right\rvert<\varepsilon$ $x<0\ \left\lvert\frac{x^2-25}{x-5}-10\right\rvert<\varepsilon \Leftrightarrow x<0\ \lvert x-5\rvert<\delta$	Precision limits, formal proofs. How to find epsilon, each field is moved and then done. Create factoring and cross out the same. P20 wants to find the x value obtained from epsilon over 5. The number 5 on the left side moves to the right, so it becomes a division.



Continuous conditions, there are conditions (i) and (ii). Condition (i) 3-t = 0. Meanwhile, 3- means t is less than 3, 3+ means t is more than 3. Because 3- and 3+ have the same value, then f(t) exists. (iii) f(t) is continuous, and f(t) is continuous because conditions (i) and (ii) are the same. If the two conditions are not the same, f(t) is not continuous.

In question 3, while P20 demonstrated procedural knowledge in solving formal proof problems, they struggled to comprehend the underlying principles of the proof. P20 did not understand that in proving limits, the epsilon value is provided, and the delta value must be determined. This lack of understanding led to an epistemological obstacle. According to Brousseau (2002), epistemological obstacles are challenges that arise from the nature of the learning approach based on the concept itself, while Fuadiah (2015) identifies epistemological obstacles as those related to the application of the concept. Finally, the conclusion regarding the learning obstacles experienced by P20 is summarized in Table 7.

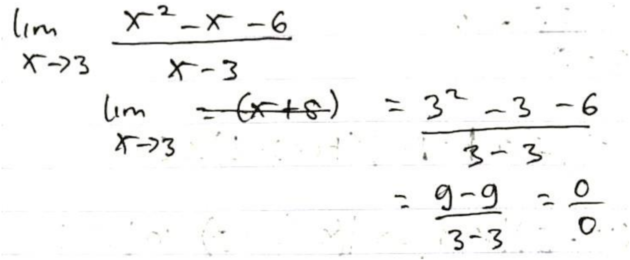
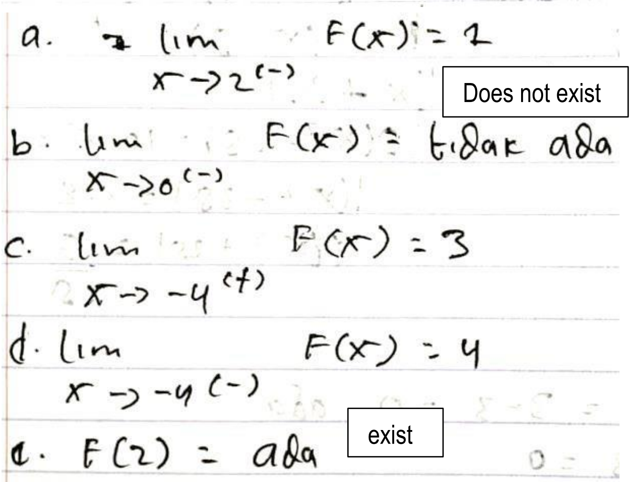
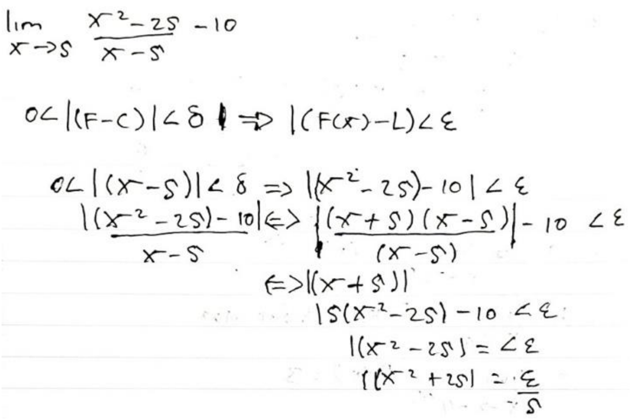
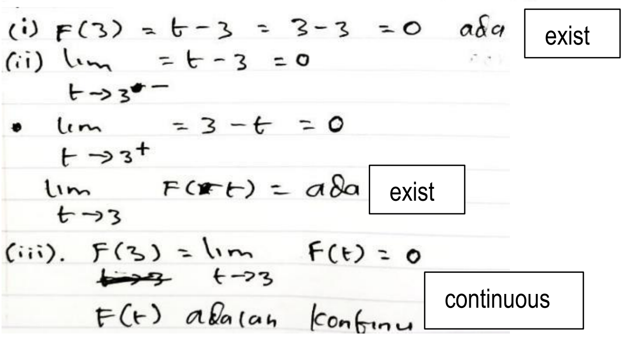
Table 7. The learning obstacles in high-ability students

Question Number	Learning Obstacle
1	×
2	epistemological obstacle
3	epistemological obstacle
4	×

Moderate-Ability Student: P22

The analysis of P22's responses to the four interview questions revealed several insights into the learning obstacles encountered as presented in Table 8. In response to Question 1, P22 did not utilize the factoring method due to feelings of uncertainty and instead opted for substitution. This choice suggests that P22 faced limitations in the prerequisite knowledge required for solving limits of functions, particularly in finding the roots of quadratic functions. These skills are considered fundamental pre-calculus concepts necessary for understanding limits. Consequently, it was concluded that P22 experienced ontogenic learning obstacles, which Puspita et al. (2023) classifies into three categories: psychological, instrumental, and conceptual. In P22's case, the hesitation in applying factoring, coupled with the inability to effectively use this method, reflects both psychological and instrumental ontogenic obstacles.

Table 8. The interview results of moderate-ability student

P22 Answer	The Conclusion of Interview Results
	P22 uses the substitution method because it cannot be factored. Even though P22 hadtried to do factoring in another paper, he was still unsure, so he decided to just makea direct substitution. Where he finds $\frac{0}{0}$ that is meaningless.
	P22 chose the full dot; on the left side is1, the answer is 1. P22 said he was in a hurry to answer, so he answered nothing. P22 continued, that the answer should be 3 because the arrow pointing to 3 is an empty point. The value is 3 because, at point -4 from the right, the empty point corresponds to 3. The value is 4 because, at point -4 from the right, the full point is 4. P22 said he forgot why he could answer yes. He doesn't yet understandwhat f(2) is looking for.
	P22 solved problem number 3 by using the formula first, namely $0 < \|f\ (x) - L\| < \ \delta$ and $0 < \|f\ (x) - L\| < \ \varepsilon$ then P22 used the factoring method but was still hesitant. The numerator and denominator are crossed out. So P22 decreases x+5. But how to write it goes back and forth.
	P22 says that (i) and (ii) are conditions that must be met. T approaching 3 from the left is 3-3 = 0. T approaching 3 from the right is also 0. Part (iii) is continuous because the positive and negative values are the same, so f(t) is continuous. If the value is positive or on the right, for example, 2, then, for example, on the left, 3 so they are different; then the function is not continuous.

In Question 2a, P22 demonstrated a misunderstanding regarding the determination of limit values, erroneously focusing on the focal points of the graph rather than applying the correct limit evaluation methods. Similarly, in Question 2b, P22 acknowledged being in a hurry when answering, resulting in an incomplete response. In Question 2c, P22 argued that the answer should be 3, associating the empty point with an arrow pointing toward 3. This suggests that P22 had a limited understanding of the graph's behavior, particularly in discerning the function's direction at specific points. Moreover, in Question 2e, P22 admitted to forgetting the reasoning behind a previously correct answer, indicating a lack of understanding regarding the meaning of $f(2)$. These difficulties reflect psychological ontogenic obstacles, primarily arising from time pressure and an incomplete conceptual grasp. Furthermore, P22's struggle to comprehend the concept of one-sided limits indicates the presence of epistemological obstacles, which are rooted in a lack of deep understanding of the underlying mathematical concepts.

In Question 3, P22 initially succeeded in applying the factoring method, though hesitantly. However, the solution was incomplete, and P22 did not manage to fully resolve the problem. This suggests that while P22 demonstrated some procedural knowledge, the inability to completely solve the problem can be attributed to epistemological obstacles, which are indicative of limitations in understanding the formal definition of limits. The hesitation observed during this process also points to psychological ontogenic obstacles, reinforcing the idea that emotional or psychological factors significantly affect P22's performance. Lastly, in Question 4, P22 did not encounter any notable learning obstacles, indicating that this particular concept or problem type was well within P22's grasp.

Based on the findings from P22's interview responses in Table 8, the following types of learning obstacles were identified in Table 9. The categorization of these obstacles aligns with the theoretical framework of ontogenic and epistemological barriers to learning, suggesting that P22's difficulties are rooted in both emotional and conceptual challenges.

Table 9. The learning obstacles in moderate-ability students

Question Number	Learning Obstacle
1	Psychological and instrumental types of ontogenic
2	Psychological, Psychological ontogenic, and epistemological obstacles
3	Psychological ontogenic and epistemological obstacles
4	×

Low-Ability Student: P17

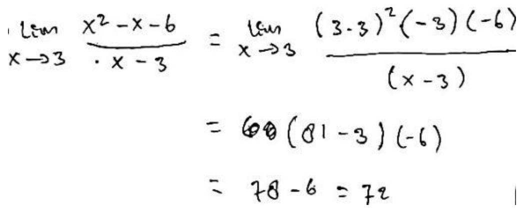
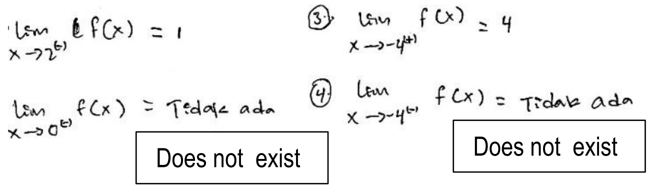
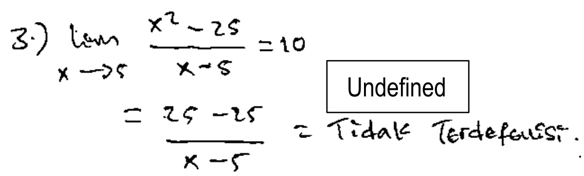
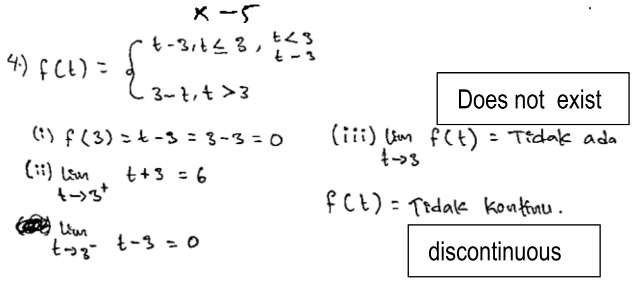
The analysis of P17's responses to the four interview questions revealed several key learning obstacles that can be classified into ontogenic and epistemological categories as summarized in Table 10. Furthermore, in Question 1, P17 struggled with factoring and opted to directly substitute into the function. This indicates a lack of mastery over the prerequisite material for solving limits, particularly the skill of factoring quadratic functions, which are foundational to understanding limits. According to Sukarma et al. (2023), students who face ontogenic obstacles encounter difficulties due to insufficient understanding of essential concepts or prerequisite material. In this case, P17's inability to apply factoring methods reflects a conceptual ontogenic obstacle.

In Question 2, P17 exhibited a limited understanding of how to determine the value of the function limit. Specifically, P17 believed that only points marked with a thick dot could be used to determine the function's limit. This misconception is an example of an epistemological obstacle. Brousseau (2002) explains that epistemological obstacles arise when students' understanding is constrained by limitations in how they interpret mathematical contexts. P17's narrow focus on the thick points without recognizing the broader concept of limits exemplifies this type of barrier.

In Question 3, P17 was unable to solve the formal proof of limits due to a lack of understanding of the appropriate methods for formal proof in the context of limits. This failure to solve the problem is attributed to epistemological obstacles, as P17's cognitive conflict arose from a disparity between the informal concept image and the formal definition of limits. Tall and Vinner (1981) assert that such cognitive conflicts occur when the concept image is inconsistent with the formal definition, which aligns with P17's difficulty in understanding formal limit proofs.

In Question 4, P17 demonstrated limited understanding regarding the meaning of the limit sign $x \to 3^+$ (right-hand limit) and $x \to 3^-$ (left-hand limit). P17's inability to interpret these symbols correctly and to understand their implications when analyzing function graphs represents another epistemological obstacle. This limitation reflects P17's incomplete grasp of one-sided limits and their role in determining the behavior of functions at specific points.

Table 10. The interview results of low-ability student

P17 Answer	The Conclusion of Interview Results
	P17 doesn't know how to do factoring. P17 also made a mistake in carrying out the operation.
	P17 looks at point 2 then at the thick point there. The 2 at that point is parallel to the number 1. Approaching 0, the answer is none because there is no bold point leading to 0, so I answered none. Choose the thick part c as well; if it's empty it means it's not there, like in part d. Part d is missing as well. Part e remains unanswered. Part e is not answered.
	P17 immediately substituted because he didn't know how to solve it with formal proof.
	P17 said that the way to solve this problem, part (i) first P17 takes from t-3, t < 3. Then part (ii) immediately adding and subtracting, P17 believes that the sign 3+ indicates addition, so the result is 6. Conversely, 3- means subtracted so it becomes 0. Part (iii), the no because the value is not the same in part (ii). P17 concluded that it was not continuously seen from the unequal values between 3+ and 3-. Then P17 was asked again what (i) and (ii) again, P22 answered that he did not understand it and only saw the example provided

	on the whiteboard during the lecturer gave him when teaching process.

Based on the findings from P17's interview responses, the following learning obstacles were identified in Table 11. These obstacles suggest that P17 faces significant challenges in mastering both foundational and advanced concepts related to limits, with epistemological barriers being the primary hindrance to progression.

Table 11. The learning obstacles in low-ability students:

Question Number	Learning Obstacle
1	Conceptual ontogenic obstacle.
2	Epistemological obstacle
3	Epistemological obstacle
4	Epistemological obstacle

In summary, the analysis revealed distinct learning obstacles experienced by students of varying abilities. High-ability students primarily encountered epistemological obstacles, which are related to limitations in understanding key concepts or their applications. Moderate-ability students faced a combination of psychological ontogenic obstacles, instrumental ontogenic obstacles, and epistemological obstacles, indicating challenges in both the application of specific methods and in conceptual understanding. Low-ability students, on the other hand, were predominantly affected by conceptual ontogenic obstacles and epistemological obstacles, particularly due to gaps in prerequisite knowledge and limited understanding of formal mathematical concepts. Given these findings, lecturers can develop targeted strategies to minimize these learning obstacles, focusing on addressing both foundational conceptual gaps and the development of critical thinking skills necessary for advanced mathematical understanding.

Implementation of Didactical Design for Teaching Limits of a Function

The implementation of an effective teaching design for the concept of the limit of a function can be developed to minimize the learning obstacles identified in this study. The design process for teaching the limit of a function follows several key steps:

1. Analysis of Learning Obstacles: The first stage involved analyzing the learning obstacles related to the limit of a function, as outlined in the results section of this article. This analysis identifies the specific challenges students face in understanding the concept of limits.
2. Review of Lecturer's Teaching Design: The second stage examined the existing calculus course design used by lecturers at Universitas Khairun. It was found that Purcell's textbook is the primary reference used, with some variations in the teaching and learning approach to suit the local context.
3. Review of Published Research: A study of relevant research articles on teaching the limit of a function provided further insights into effective teaching strategies.
4. Expert Validation: The proposed didactical design was then validated through a Focus Group Discussion (FGD) held on September 28, 2024, involving subject matter experts.

Based on the findings from these stages, the recommended teaching phases for the limit of a function have been formulated, as depicted in Figure 4.

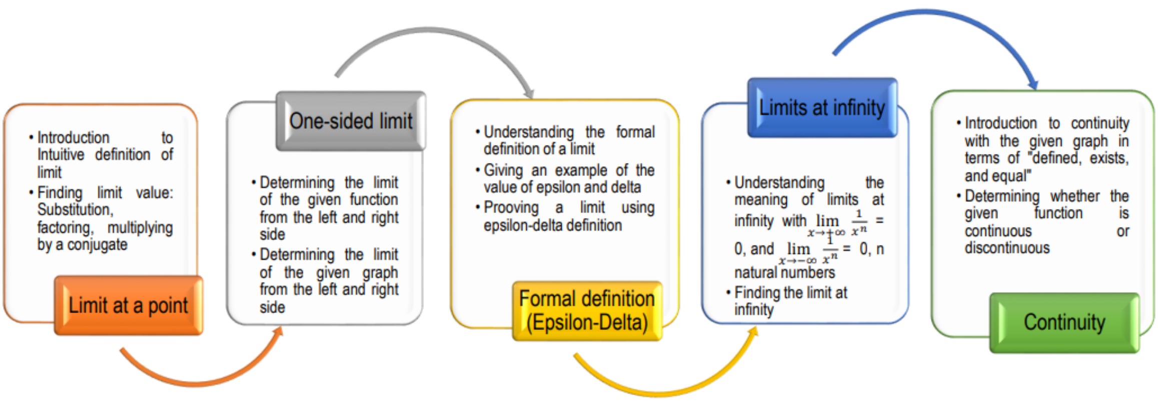


Figure 4. Teaching phase based on the content of the limit of function

The learning obstacles associated with determining the limit value typically arise when students are unfamiliar with factoring. In such cases, students often resort to direct substitution. To address this, the teaching design begins by introducing the concept of the limit intuitively (Oehrtman, 2008). Students are initially presented with a function where substitution leads to an indeterminate form, thereby illustrating the need for alternative methods, such as factoring or multiplying by the conjugate (Hardy, 2009; Nagle, 2013). Once students grasp the intuitive meaning of a limit and learn how to determine the limit using these methods, they progress to the next phase: one-sided limits.

The one-sided limit phase is aimed at determining the limit of a function as it approaches a given point from one side only. During this phase, students are provided with both the function and its graph and are asked to compute the left and right limits. An example of this didactical design is presented in Figure 5.

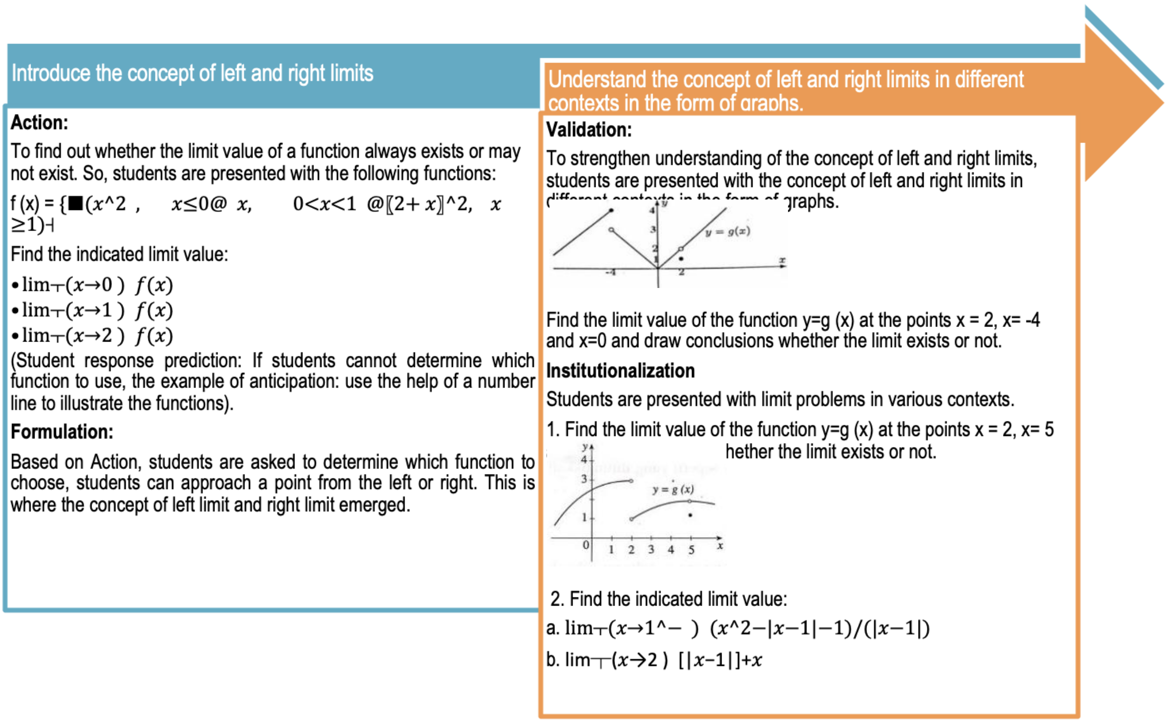


Figure 5. Example of didactical design for one-sided limits

Once students have learned how to determine whether the limit of a function exists, the next phase of instruction focuses on the formal limit. A significant learning obstacle in this phase is that students often focus solely on procedural aspects of limit proofs without understanding the conceptual foundation, particularly the epsilon-delta definition. This finding aligns with previous research by Nagle et al. (2017). To address this, teaching begins by introducing the formal definition of limits, followed by practical examples that illustrate the values of epsilon and delta. This approach helps students visualize these values as real numbers, thus facilitating a deeper understanding of the formal limit definition.

Additionally, the teaching design incorporates limits at infinity, which is an essential extension of the concept of limits. Although this phase was not included in the initial stages of the research due to its exclusion from previous coursework, it will be addressed in future instructional phases. The focus of this phase is to explore the behavior of a function as x approaches infinity. The final teaching phase is dedicated to determining whether a function or its graph is continuous or discontinuous. Through this phase, students learn how to analyze the continuity of functions, building on the knowledge they have acquired in previous phases.

The diagnostic tests, which were administered to two student groups (Group 1 and Group 2), provide an important measure of the effectiveness of the didactical design. These tests were given at the start and end of the teaching intervention to assess students' understanding of the limit of a function.

The results of the diagnostic test, displayed in Figure 6, show a marked reduction in learning obstacles in Group 2 after the implementation of the didactical design. The most prominent learning obstacles remaining in this group were related to the formal definition of limits and limits at infinity. Consequently, it is recommended that calculus instructors place greater emphasis on these subtopics to ensure students develop a comprehensive understanding.

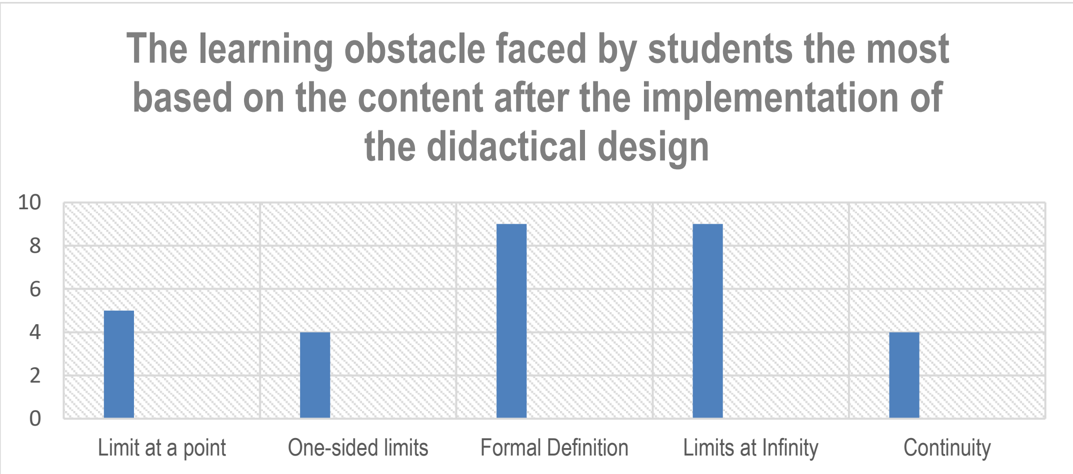


Figure 6. The learning obstacles faced by students the most in group 2 based on the content

Finally, the implementation of the didactical design led to a significant reduction in the number of learning obstacles compared to the pre-intervention phase, as shown in Figure 3. However, the results also highlight that challenges persist in the areas of formal definitions and limits at infinity, suggesting that lecturers should provide additional support and clarification in these areas. By addressing these remaining obstacles, lecturers can further enhance students' understanding of limits and improve their overall learning experience.

CONCLUSION

This study led to three significant conclusions regarding the learning obstacles in the concept of limits in calculus. First, the research revealed that among the four sub-contents of limits of function—namely limit at a point, one-sided limit, formal definition, and continuity—the most prevalent learning obstacle was encountered in understanding the formal definition. This highlights the importance for educators to prioritize the teaching of formal definitions, particularly the ε-δ definition, and to ensure that students grasp the conceptual understanding needed to prove the existence of limits. Second, the study found that learning obstacles varied according to the students' ability levels. High-ability students predominantly faced epistemological obstacles, while moderate-ability students encountered psychological ontogenic, instrumental ontogenic, and epistemological obstacles. Low-ability students primarily experienced conceptual ontogenic and epistemological obstacles. The high-ability students demonstrated a procedural understanding of the formal definition but struggled with its deeper meaning. Moderate-ability students tended to use substitution methods instead of correctly applying factoring techniques, and they also struggled with conceptualizing the formal definition. Low-ability students exhibited significant challenges in understanding any of the four sub-content areas due to insufficient prerequisite knowledge, such as factoring and solving inequalities.

Despite the positive impact of the didactical design implemented in this study, which contributed to reducing learning obstacles and enhancing comprehension of the limit of function, certain limitations were observed. While the occurrence of learning obstacles decreased, they were not completely eradicated. This suggests that while the didactical design approach showed promise, further efforts are required to address the persistent barriers that hinder full comprehension. Additionally, the study's findings underline the need for continuous refinement of teaching strategies and materials to ensure that these obstacles are minimized more effectively.

Based on the results and limitations of this research, future studies should focus on exploring more advanced and comprehensive methods to further reduce learning barriers in the teaching of limits and other calculus topics. Research could investigate innovative teaching strategies, including adaptive learning techniques or technology-enhanced learning tools, to better support students across different ability levels. Moreover, further research should examine the broader application of didactical designs to other areas of mathematics education, exploring their effectiveness in various instructional contexts to improve overall student understanding and reduce epistemological and conceptual obstacles.

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Declarations

Author Contribution	:	WST: Conceptualization, Writing - Original Draft, Project Administration, and Visualization. DS: Conceptualization, Formal Analysis, and Methodology. ECM: Formal Analysis, Validation, and Supervision. K: Formal Analysis, Validation, Supervision, Writing – Review and Editing.
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