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Analysis of pre-service mathematics teachers' proof comprehension through Toulmin's argumentation model

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Abstract

The comprehension of mathematical proofs by preservice mathematics teachers is vital for their ability to effectively teach mathematical reasoning. Despite its importance, existing research reveals a significant gap in preservice teachers' understanding and application of formal proof methods, especially in the context of mathematical argumentation. This study examined how preservice teachers construct mathematical proofs, using Toulmin's argumentation model as a framework. A qualitative exploratory case study design was adopted, involving written proofs from 72 third-year preservice teachers at a South African university, supplemented by task-based interviews with nine participants. The findings indicate that 62.5% of the participants were able to construct correct direct proofs, and 61.1% applied the contraposition proof method correctly. However, only 30.6% produced valid proofs using the contradiction method. Further analysis uncovered notable gaps in essential components of proof construction, such as warrants, backing, and rebuttals, particularly when dealing with tasks requiring contraposition and contradiction methods. While many participants (62.5%) demonstrated procedural fluency in direct proofs, 31.9% failed to provide explicit definitions or logical precision, suggesting a superficial engagement with proof construction. These results highlight the need for teacher education programs to emphasize a deeper conceptual understanding of proof structures, which is crucial for preparing preservice mathematics teachers to foster reasoning and argumentation skills in their future classrooms.

Keywords: Contradiction, Contraposition, Direct Proofs, Preservice Mathematics Teachers, Proof Comprehension, Toulmin's Argumentation Model

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Over the years, curriculum frameworks globally have emphasized the importance of developing reasoning and proof skills as essential components of mathematics education. Frameworks such as the Australian Curriculum (2019), the Ontario Ministry of Education (Canada, 2020), and the Department for Education (United Kingdom, 2021), among others, consistently highlight reasoning and proof as fundamental elements in secondary school mathematics. This emphasis is also reflected in the work of prominent scholars in mathematics education (e.g., Brodie, 2010; Goos & Kaya, 2020; Knuth, 2002b; Selden & Selden, 2017; Stylianides et al., 2017, 2024; Varghese, 2009), who argue that reasoning and proof are essential for developing a deep understanding of mathematical concepts. Similarly, the process standards set forth by the National Council of Teachers of Mathematics (NCTM, 2000) advocate for students to formulate and test conjectures, construct and assess mathematical arguments, and apply a variety of reasoning strategies and proof techniques.

Although South Africa's Curriculum and Assessment Policy Statement (CAPS) (Department of Basic Education, 2011) does not explicitly prioritize "reasoning and proof," it encourages students to engage with proof techniques, emphasizing the importance of teaching the rationale behind proofs. These practices play a crucial role in promoting conceptual understanding and equipping students with the critical thinking and problem-solving skills necessary for academic achievement and success in real-world contexts.

This recognition underscores the necessity for secondary school mathematics teachers to possess a solid understanding of mathematical concepts, particularly in proof construction, as teachers cannot effectively teach topics or concepts they do not fully comprehend (Mukuka & Alex, 2024). In the South African Further Education and Training (FET) phase, the proof of prescribed theorems and the derivation of formulas are integral components of routine instructional practices (Department of Basic Education, 2011). Research by Brodie (2010) and Maoto et al. (2018) further emphasizes that reasoning and proof should be central to classroom discussions, irrespective of the specific mathematical topic or concept being studied. This highlights the importance of providing preservice mathematics teachers with more opportunities to deepen their understanding of proofs, especially those pertinent to the curriculum they will eventually teach. However, despite these calls for an increased focus on proofs, there is a notable lack of research documenting the comprehension of proofs by preservice teachers, particularly within the context of sub-Saharan Africa, and specifically South Africa.

This study addresses this gap by investigating how preservice mathematics teachers at a South African university conceptualize and apply various methods of proof when tasked with validating mathematical statements involving fundamental number concepts. The use of proof methods, whether direct or indirect, is crucial for preservice teachers as these methods form the foundation of logical reasoning and problem-solving (Courant & Robbins, 1996, as cited in Haavold et al., 2024).

In the context of this study, we define proof as a structured reasoning process that enables learners to validate mathematical statements through logical inference (Hanna, 2020; Hanna & Barbeau, 2010). Building on Duval's (1995, as cited by Pedemonte, 2007) framework, argumentation is understood as the reasoning process that preservice teachers use when constructing proofs. This process involves establishing logical connections and justifications. While argumentation aims to convince or persuade through reasoning—whether formal or informal—proof, by contrast, emphasizes the rigorous, logical validation of a statement. A step in a proof refers to each individual component or assertion that contributes to the overall argument, ensuring that every part of the reasoning is clearly articulated and logically sound.

Direct proof is a method of demonstrating the truth of a mathematical statement by starting with known premises and using logical steps to arrive at a conclusion (Cupillari, 2024). While there may be other types of direct proofs, this study employs a proof by construction, a direct proof method that involves proving statements such as $P \Rightarrow Q$ directly (Ferry, 2010). An example of a direct proof may involve proving that the sum of two odd numbers is always even. The first step in applying proof by construction would be to let the two odd numbers be represented as $2\mathrm{m}+1$ and $2\mathrm{n}+1$, where m and n are integers. Secondly, adding them followed by factoring out 2 results in $2(\mathrm{m}+\mathrm{n}+1)$, which is clearly an even number, as it is divisible by 2.

On the other hand, indirect proofs, including proof by contraposition and proof by contradiction, approach a proof task from a different angle. In a contrapositive proof, instead of proving the original statement $P \Rightarrow Q$, one proves its equivalent contrapositive $\neg Q \Rightarrow \neg P$ (Antonini & Mariotti, 2008). An example of a contrapositive proof is showing that if the square of an integer is even, then the integer itself must be even. Rather than proving the original statement directly, we assume the contrapositive—that if an integer is odd, then its square is odd—and then verify this claim to conclude the truth of the original statement. Proof by contradiction, on the other hand, assumes the negation of the initial statement and seeks to show that this assumption leads to a logical contradiction, thereby confirming the original statement as true. A classic example involves proving the irrationality of $\sqrt{2}$. By assuming $\sqrt{2}$ is rational, one can prove a contradiction, ultimately proving its irrationality (Quarfoot & Rabin, 2022).

For preservice teachers, developing a deep understanding of proof methods and their application is essential for both their personal mathematical growth and their future effectiveness as educators. Research indicates that teachers with a solid grasp of proof can elucidate abstract concepts, correct common misconceptions, and cultivate classrooms where reasoning takes precedence over rote memorization (Lesseig et al., 2019; Mukuka et al., 2021; 2023). However, despite the critical role proofs play in mathematics education, many preservice teachers face challenges in constructing and understanding them, often resorting to memorizing steps without fully comprehending the underlying logic (Buchbinder & McCrone, 2023; Doruk & Kaplan, 2015; Moore, 1994). This reliance on rote learning inhibits the development of the critical thinking skills necessary for effective mathematical reasoning. These difficulties not only impact the future teaching practices of preservice teachers but also the mathematical competence of their students (Harel & Sowder, 1998; Knuth, 2002a).

Thus, this study seeks to analyze preservice mathematics teachers' understanding of proofs. Focusing on direct, contraposition, and contradiction methods of proof, the study aims to examine the reasoning processes that preservice teachers employ and the challenges they face when applying these methods. Toulmin's Argumentation Model provided the framework for analyzing these reasoning processes. This model is particularly valuable for identifying and assessing the key components of arguments—such as claims, data, and warrants—which are central to understanding proof comprehension. To achieve this objective, the following research questions were addressed:

1. How do preservice teachers justify their reasoning when constructing proofs for mathematical statements on number concepts?
2. What gaps or inconsistencies in Toulmin's argumentation components emerge in preservice teachers' proofs, and how do these reflect their conceptual understanding?

As previously mentioned, this study is grounded in Toulmin's Argumentation Model, which offers a structured framework for analyzing the reasoning and logic that underpin arguments. Introduced by Stephen Toulmin in The Uses of Argument (Toulmin, 1958; 2003), the model identifies six key components that are essential for constructing a coherent and robust argument: claim, data, warrant, backing, qualifier, and rebuttal. The claim represents the conclusion or assertion being put forward, which is supported by data—evidence or facts that substantiate the claim. The warrant serves to link the data and the claim, providing the logical reasoning that connects them. The backing further strengthens this connection by offering additional support for the warrant's validity. The qualifier specifies the degree of certainty associated with the claim, while the rebuttal acknowledges potential counterarguments or exceptions.

Applying Toulmin's model to mathematical proof allows for a detailed analysis of reasoning processes, which enables teachers to identify gaps in students' logical structures and reasoning. Using the earlier example of proving that the sum of two odd numbers is always even, the claim asserts that adding two odd integers results in an even number. The data supporting this claim involves representing odd numbers as $2m + 1$ and $2n + 1$, where m and n are integers. As indicated earlier, adding these expressions produces something which is clearly divisible by 2. In this case, the warrant would connect this data to the conclusion by appealing to the definition of even numbers. At the same time, the backing draws upon established properties of integers and arithmetic operations. This means that common issues in students' proofs, such as omitting the specification that m and n are integers, can weaken the argument by creating logical ambiguities.

Similarly, in the proof of the irrationality of the $\sqrt{2}$, Toulmin's model provides a lens through which to analyze the logical steps involved. The claim is that $\sqrt{2}$ cannot be expressed as a ratio of two integers. To support this, the data assumes the contrary, that $\sqrt{2} = \frac{a}{b}$, where a and b are coprime integers, $b \neq 0$. The proof proceeds by showing that this assumption leads to a contradiction. If the initial assumption of coprimality is violated, then the contradiction is reached. In this sense, the warrant connects the assumption of rationality to the logical inconsistency that arises, while the backing includes the properties of divisibility and integers. This proof is valid within the context of integer arithmetic and rational numbers, as it assumes the definition of rationality as a fraction in simplest form. However, omitting critical assumptions, such as a and b are coprime integers and $b \neq 0$, could weaken the data and lead to illogical arguments.

Toulmin's model has been effectively employed in various studies to analyze mathematical reasoning and proof. For instance, Pedemonte (2001) explored the transition from argumentation to formal proof, emphasizing the role of warrants in bridging informal and formal reasoning. Tchonang Youkap et al. (2019) utilized Toulmin's framework to analyze secondary school students' geometric proofs, revealing frequent gaps in warrants and backing. Similarly, Knipping (2008) examined students' proofs in algebra, identifying recurring issues in their argumentation structures. In another study, Rodríguez-Nieto et al. (2023) integrated Toulmin's model with the Extended Theory of Mathematical Connections to analyze an episode focused on proving the continuity theorem for functions of a real variable. Their findings highlighted the importance of connections between mathematical concepts, theorems, and properties in constructing coherent and logically supported arguments. These studies demonstrate the model's effectiveness in pinpointing specific areas where students face difficulties and offer valuable insights for targeted instructional interventions. By applying Toulmin's model to analyze preservice teachers' proofs, this study contributes to a deeper understanding of the strengths and weaknesses in their reasoning processes, thus providing a foundation for designing pedagogical strategies aimed at improving preservice teachers' proof comprehension.

METHODS

Study Design and Context

This research employed a qualitative exploratory case study design to examine preservice teachers' comprehension of proof techniques. The design facilitated the collection and analysis of data aimed at understanding how preservice teachers interpret and apply various proof methods. The participants were third-year preservice mathematics teachers enrolled in a Bachelor of Education program at a South African university. Unlike their peers in pure or applied mathematics programs, these students did not engage in intensive proof-oriented coursework. Instead, their curriculum focused on mathematics education and pedagogical approaches relevant to their future teaching careers. Prior to this study, participants had completed an introductory module on real analysis, which provided foundational exposure to proof construction and logical reasoning. The module covered topics such as truth tables for quantifying statements, the application of various proof methods, and reasoning techniques for validating mathematical propositions, including the characterization of the real number line as an ordered field. The curriculum also explored problem-solving strategies and theorem proofs, as well as sequences, series, sets, and boundedness in the context of real analysis. These topics provided a conceptual basis for understanding proof methods, though the level of rigor was designed for prospective educators rather than professional mathematicians.

In this study, participants were tasked with answering three proof-based questions designed to assess their proficiency with direct (proof by construction) and indirect (proof by contraposition and contradiction) methods for justifying the validity of selected mathematical statements. The study was motivated by recurrent challenges identified in the literature regarding preservice teachers' difficulties with proof comprehension (e.g., Mukuka et al., 2019). The three proof questions administered to the participants are illustrated in Figure 1.

Item 1: Show that the sum of any two odd numbers is always even

Item 2: For all integers m and n, prove that if the product mn is even, then either m is even or n is even.

Item 3: Prove that $\sqrt{3}$ is an irrational number

Figure 1. Administered proof tasks

While completing the tasks shown in Figure 1, participants were asked to select and apply appropriate proof methods, such as direct proof, proof by contraposition, or proof by contradiction. In the first task, participants were expected to recognize that direct proof by construction was the most suitable approach. This task required expressing odd numbers in a general form and utilizing basic arithmetic to demonstrate that their sum is always even. A strong understanding of algebraic manipulation and the properties of odd and even numbers were necessary for success in this task.

For the second task, participants were required to prove that if the product of two integers is even, then at least one of the integers must be even. Both contraposition and contradiction methods were applicable here. In using proof by contrapositive, participants would restate the problem as "If neither m nor n is even, then their product is odd" and proceed to show that this restatement is logically consistent with the given conditions. Alternatively, proof by contradiction would involve assuming that both m and n are odd and demonstrating that this assumption contradicts the given statement.

The third task asked participants to prove that $\sqrt{3}$ is irrational, which is typically done using proof by contradiction. Participants would assume that $\sqrt{3}$ is rational and express it as a fraction in its lowest terms. They would then engage in a sequence of logical deductions, each justified by established mathematical principles, ultimately arriving at a contradiction to demonstrate that $\sqrt{3}$ cannot be rational. These tasks were designed to evaluate participants' understanding of proof methods and their ability to select appropriate techniques based on the nature of the mathematical statement. Our aim was to assess both their final conclusions and the reasoning behind their choice and application of proof methods.

Participant Selection

Out of 189 submissions, 72 scripts were selected for analysis based on the principle of data saturation, which indicated that further analysis would not yield new insights (Fusch & Ness, 2015). This decision was informed by a systematic review of the written scripts, wherein an iterative analysis of smaller batches identified recurring themes, common misconceptions, and typical reasoning patterns. The selected scripts were labeled from S001 to S072 for identification purposes.

In addition to the script analysis, task-based interviews were conducted with nine preservice teachers, purposively selected to ensure a diverse representation across different response categories (correct, partially correct, and incorrect). Three participants were chosen randomly from each category to provide a balanced perspective on their reasoning processes, proof method application, and the challenges they encountered. This purposive sampling strategy ensured that the interview data complemented the script analysis, thereby enhancing our understanding of preservice teachers' proof comprehension. The task-based interviews did not follow a predetermined interview guide. Each participant was asked to select one of the proof questions displayed in Figure 1, given a piece of paper, and instructed to construct a proof. Following the construction of their proof, we asked questions to solicit further explanation of their reasoning.

Data Analysis

Written script analysis was conducted using content analysis, guided by Toulmin's model of argumentation (Toulmin, 1958; 2003). This framework allowed for a structured assessment of the key components in participants' proofs, including claims, data, warrants, backings, qualifiers, and rebuttals. These components provided a basis for evaluating the appropriateness of the chosen proof methods, the handling of mathematical statements, the logical progression of arguments, the identification of common errors or misconceptions, and the overall clarity of explanations. Table 1 presents the analytical categories and guiding questions derived from Toulmin's model to direct the data extraction and analysis.

Table 1. Predetermined themes and guiding questions for data extraction and analysis

Analysis Code	Guiding Question
Claim	Is the mathematical statement being proven clearly articulated?
Data	Is the selected proof method appropriate? Are the definitions, assumptions, and given information explicitly stated?
Warrant	Is there a clear and logical connection between the data and the claim?
Backing	Are additional justifications or properties used to support the warrant?
Qualifier	Are there any indications of generality or limitations in the argument?
Rebuttals	Are potential counterarguments addressed, or alternative assumptions considered?

Content analysis was employed to systematically categorize the responses into four groups: blank, incorrect, partially correct, and correct. This classification enabled the quantification of preservice teachers' performance and facilitated the identification of patterns in their approach to proof. Previous studies (Tatira, 2021, 2023; Tatira & Mukuka, 2024) employing similar response categorization alongside Toulmin's model provided a comparative framework for analyzing our findings and drawing connections with other research on proof comprehension.

Additionally, data from task-based interviews were analyzed thematically to explore participants' reasoning processes in greater depth. This analysis focused on the claims participants aimed to prove, the data and warrants they used, and the challenges they faced in applying proof methods. Thematic coding was conducted to categorize participants' responses based on their articulation of claims, use of data and warrants, reliance on backing, and handling of qualifiers or rebuttals. This approach offered insights into whether participants relied on procedural familiarity or demonstrated a deeper conceptual understanding in proof construction. Thematic analysis also illuminated tendencies toward memorization versus engagement with the logical structure of proofs.

Ethical Clearance

The study adhered to ethical standards, ensuring participant privacy and confidentiality. No identifying information was disclosed in the data dissemination process, and participants provided informed consent for the use of their written and oral responses for research purposes. This study forms part of an ongoing research project approved by the Institutional Ethical Review Board of the university, with the authors and participants affiliated with the institution. The ethical approval number is FEDSRECC01-12-20, and the approval certificate permitted the study to be conducted with preservice teachers in their second or third year. The broader research project aims to profile preservice teachers' content knowledge in various mathematical concepts, with the goal of enhancing their mathematical knowledge for teaching.

RESULTS AND DISCUSSION

This study aimed to examine the understanding of proof methods among preservice teachers and their application in validating mathematical statements related to number concepts. The results are presented in two sections: Analysis of Written Scripts and Insights from Task-Based Interviews. The first section evaluates preservice teachers' responses to three proof tasks using Toulmin's model of argumentation, focusing on key components such as claim, data, warrant, backing, qualifier, and rebuttal. However, it is important to note that not all six components of Toulmin's model were explicitly articulated in the preservice teachers' solutions, which may reflect gaps in their reasoning or presentation. The second section discusses the preservice teachers' thought processes during task-based interviews, reflecting on their written solutions and addressing challenges or misconceptions encountered. Additionally, connections are drawn between their interview responses and the patterns observed in their written work.

Analysis of Written Proof Scripts

The results summarized in Table 2 highlight distinct patterns in preservice teachers' reasoning processes and areas of difficulty. Each task is analyzed individually, with excerpts from participants' written work providing further insights. Responses are categorized into four groups: left blank, incorrect, partially correct, and correct.

Table 2. Response categories for all Items

Response Category	Item 1		Item 2		Item 3	
	Count (n)	*%*	*Count (n)*	*%*	*Count (n)*	*%*
Left blank	4	5.6	3	4.2	1	1.4
Incorrect	7	9.7	10	13.9	0	0
Partially Correct	16	22.2	15	20.8	49	68.0
Correct	45	62.5	44	61.1	22	30.6
Total	72	100	72	100	72	100

Item 1: Sum of Two Odd Numbers Being Even

The results in Table 2 indicate that a majority of preservice teachers (62.5%) provided correct proofs for Item 1, which suggests a relatively strong grasp of direct proof methods. The claim of this task—"the sum of any two odd numbers is always even"—was correctly stated by most participants. However, several responses lacked precision in the data, as some participants failed to specify that k in the expression $2k + 1$ must be an integer, as shown in Figure 2. This omission weakened the logical foundation of the proof, though it did not always invalidate the procedural steps. This indicates that the warrant was either incomplete or flawed, as the reasoning lacked a critical assumption necessary to establish the connection between data and the claim.

The warrant, which connects the data (definitions of odd numbers) to the claim (resulting even sum), was frequently implied rather than explicitly stated. For example, some participants did not justify why adding $2k + 1$ and $2m + 1$ results in $2\ (k + m + 1)$, a form divisible by 2. These gaps suggest an overreliance on procedural fluency without a deep conceptual understanding. Furthermore, the backing, such as justifications for integer properties, was often absent, indicating a lack of depth in reasoning.

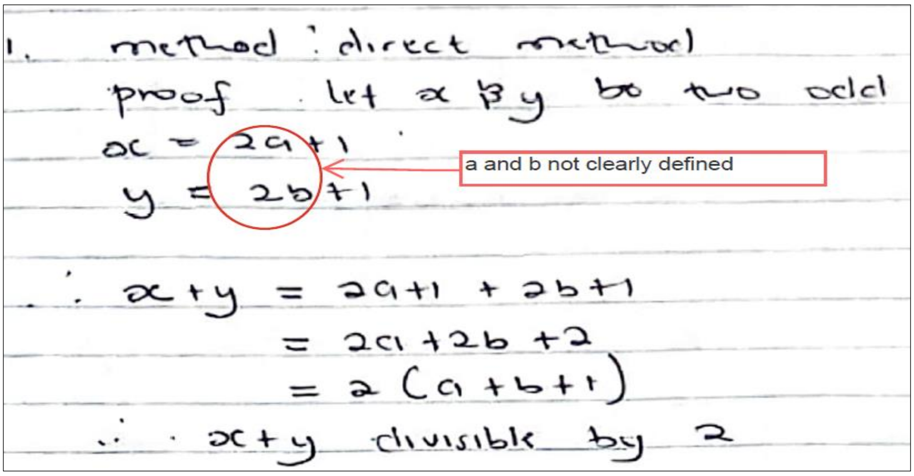

Figure 2. Solution by S008 displaying undefined variables for Item 1

Partially correct (22.2%) and incorrect (9.7%) responses for Item 1 often involved incomplete or erroneous warrants. For instance, Figure 3 demonstrates a case where the student misunderstood the relationship between odd and even numbers, leading to an invalid conclusion.

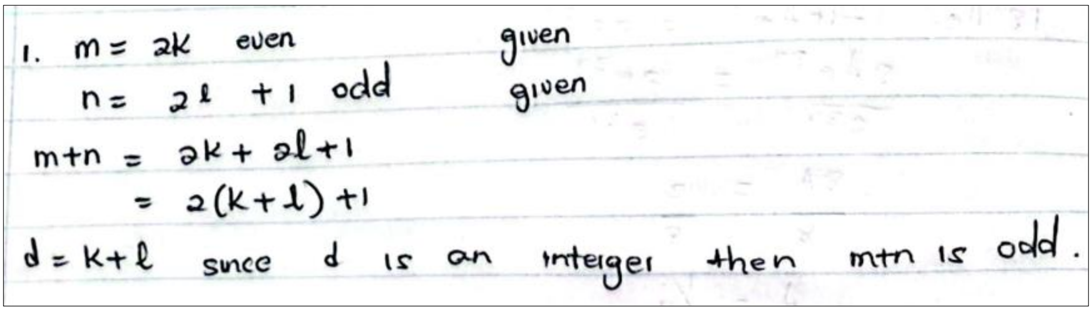

Figure 3. Item 1 solution by S021 showing incorrect assumptions or misunderstanding of a claim

In addition to misunderstandings of assumptions and failure to grasp the claim, the solution in Figure 3 also illustrates unclear variable specifications, such as the omission of stating that *k* and *l* are positive integers. Another common issue was confusion regarding proof methods, such as direct proof, contraposition, or proof by contradiction. Some participants mixed up these methods, resulting in incomplete or incorrect solutions. In some cases, participants identified one proof method but inadvertently employed another, suggesting a lack of understanding of the distinctions between the methods or simple oversight. Figure 4 shows an example of such a mismatch.

In the case of Figure 4, the solution reflects an incomplete or missing warrant, as the logical reasoning connecting the data to the claim is neither clearly stated nor correctly aligned with the chosen proof method.

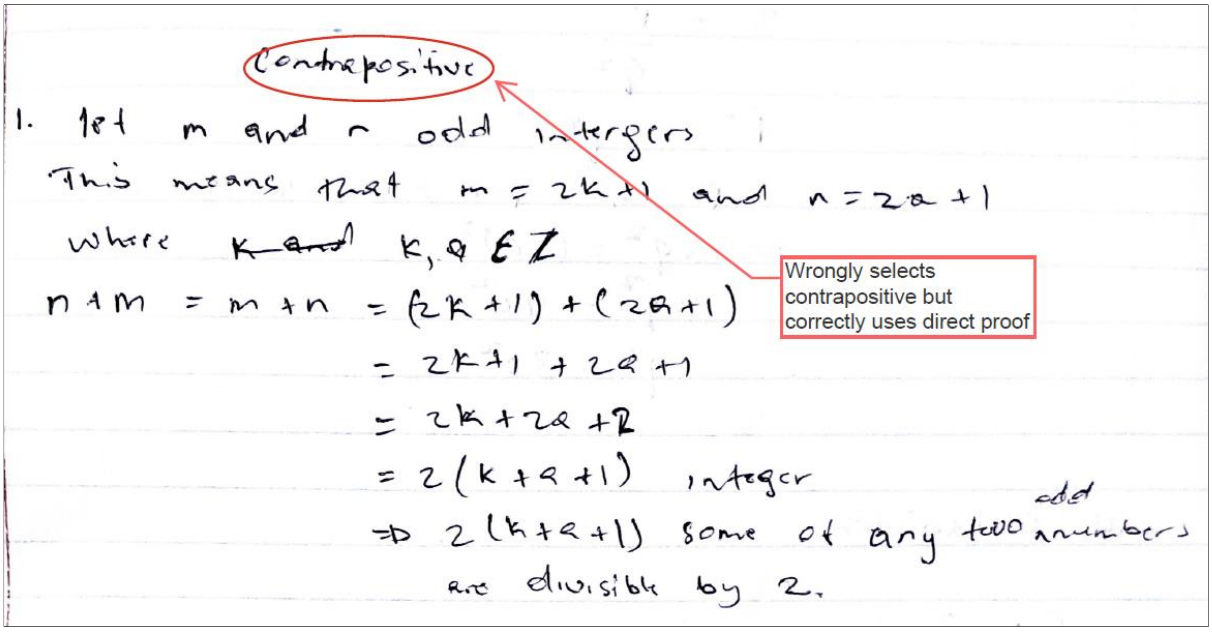

Figure 4. Solution by S004 displaying a mismatch between selected and implemented proof methods for Item 1

Item 2: Contrapositive or Contradiction Proof for an Even Product

Item 2 required preservice teachers to prove that if the product mn of two integers is even, then either m or n must be even. As indicated by the results in Table 2, 61.1% of responses were correct, reflecting a basic understanding of the claim and the appropriate proof methods. However, Figure 5 demonstrates that some participants in the partially correct category (20.8%) and others in the incorrect category (13.9%) struggled with selecting and applying the correct proof method.

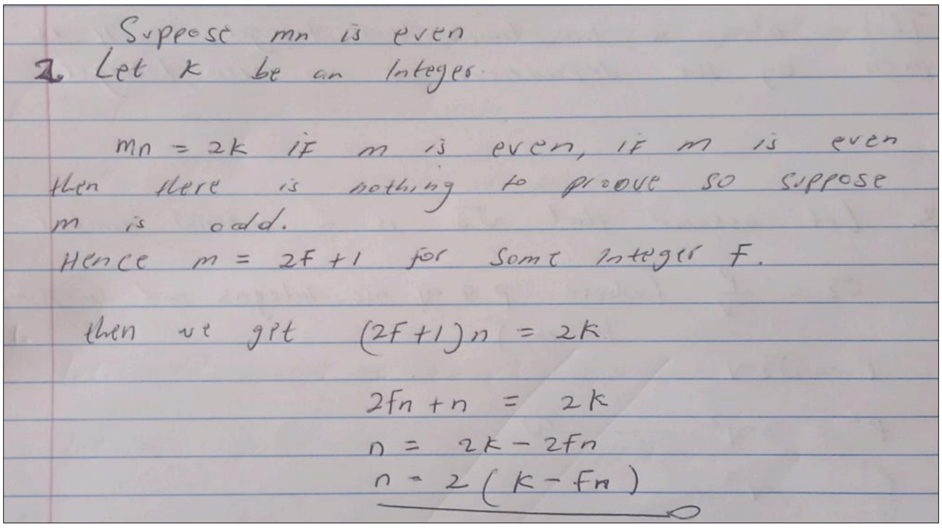

Figure 5. Item 2 solution by S031 showing a lack of coherent logical steps in a proof

The solution presented by participant S031 in Figure 5 exhibits significant gaps in reasoning and logical structure. Although the participant attempts to prove that "if mn is even, then either m or n is even," the claim is not explicitly stated, leading to ambiguity and undermining the clarity of the argument. The use of the equation $mn = 2k$, while correct in indicating that mn is even, fails to fully utilize the definitions of odd and even numbers. Furthermore, the equation $n = 2(k - fn)$ is problematic due to its circular dependency, which impedes the isolation of n as an even number and weakens the proof. Essential specifications, such as explicitly stating that k, f and n are integers, are also missing, undermining the foundational reasoning. The warrant, which should connect the assumption that m is odd to the conclusion that n is even, remains underdeveloped.

Similarly, Figure 6 provides an example where the participant's reasoning lacks a clear warrant, as the logical connection between the initial assumption and the conclusion is inadequately articulated. The introduction of new assumptions after the conclusion is reached undermines the validity of the proof and indicates a failure to recognize that a contradiction should arise solely from the original assumption and the established logical steps.

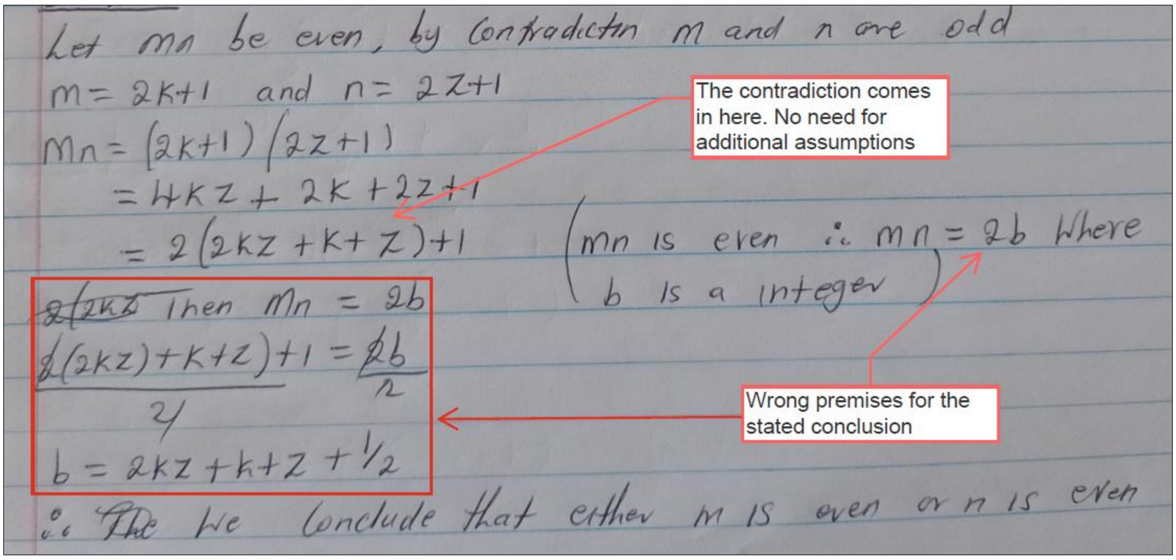

Figure 6. Item 2 Solution by S022 Displaying a Misunderstanding of Underlying Proof Structure

Item 3: Proof by Contradiction for the Irrationality of $\sqrt{3}$

Item 3 presented the most significant challenge, with only 30.6% of participants producing correct proofs. Most responses (68.0%) were partially correct, revealing widespread difficulties with the underlying logic of proof by contradiction. The claim that $\sqrt{3}$ is irrational was frequently misunderstood or ambiguously stated, as illustrated by participant S002's solution in Figure 7.

The solution presented in Figure 7 reflects a surface-level understanding in which a participant correctly assumed that $\sqrt{3} = \frac{a}{b}$, but failed to define what a and b represented or why the assumption was necessary. This omission highlights a missing data component, as the lack of explicitly stated definitions and conditions (e.g., a and b being integers) undermines the foundation of the proof. Additionally, the participant's failure to explicitly state that a and b are coprime, and that $b \neq 0$, further weakens the logical framework by neglecting critical backing, additional justifications that would reinforce the assumptions and the progression of the argument.

The proof also exhibits a flawed or incomplete warrant, as the participant did not establish a clear connection between the assumption $\sqrt{3} = \frac{a}{b}$ and the contradiction that should logically follow. Without properly specifying the relationships and properties of a and b, the reasoning process remains disconnected, resulting in logical gaps. Such inadequacies were quite prevalent and revealed participants' reliance on procedural familiarity without fully grasping the conceptual underpinnings necessary for constructing a valid and coherent mathematical proof for the task presented in Item 3.

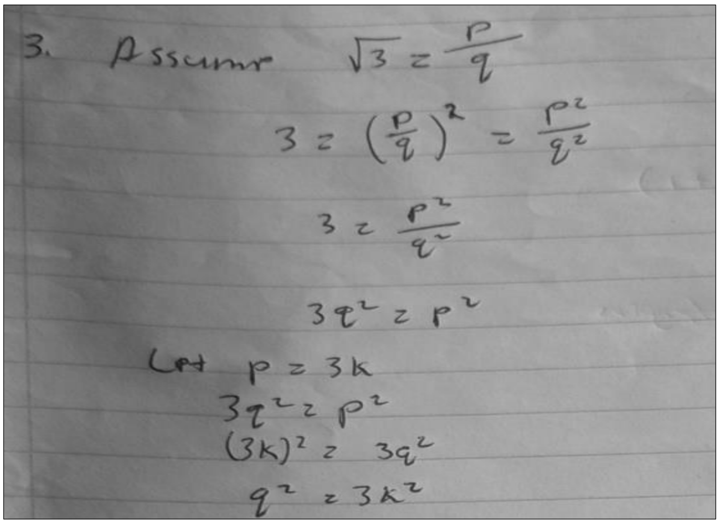

Figure 7. Item 3 solution by S002 showing incomplete assumptions, and missing variable definitions, and conclusion

Figure 8 illustrates another example where the participant claimed to use proof by contraposition but instead employed proof by contradiction. The solution contains undefined variables and lacks a proper conclusion, further weakening the argument.

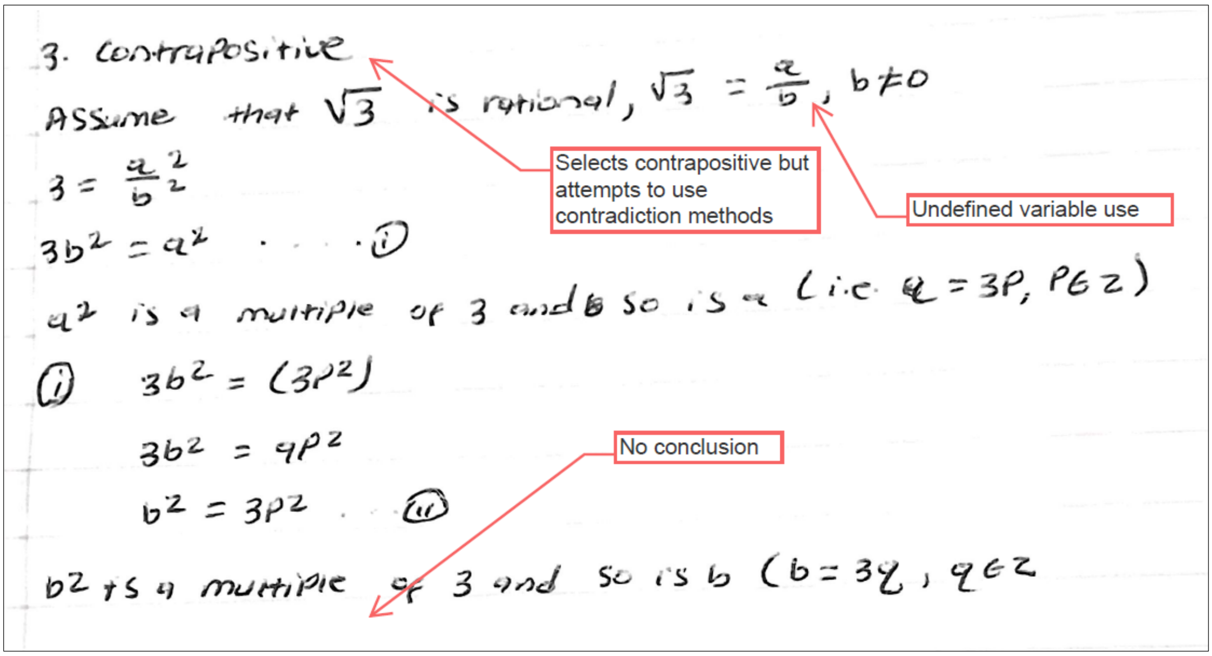

Figure 4. Mismatch between selected and implemented proof methods by S005 for Item 3

In summary, the findings reveal that while many preservice teachers provided correct responses (averaging 51.5%), significant challenges remain in their conceptual understanding, particularly when constructing valid proofs. Item 3, which required more complex reasoning, had a particularly high rate of partially correct responses (68.0%), with only 30.6% of participants successfully constructing a valid proof. The observed patterns of confusion regarding mathematical definitions, proof methods, and variable representation suggest a reliance on procedural understanding rather than conceptual insight. This highlights the need for deeper internalization of proof concepts among preservice teachers.

Insights from Task-Based Interviews

The task-based interviews provided deeper insights into the difficulties preservice teachers faced when constructing valid proofs for mathematical statements. These interviews offered participants the opportunity to reflect on their written solutions, discuss the challenges they encountered, and clarify misconceptions that arose during the process. Specifically, the interviews focused on how participants selected proof methods for the tasks at hand and the difficulties they experienced. Their responses also shed light on whether they relied on memorization or demonstrated a genuine mastery of proof construction.

In terms of selecting proof methods, it became evident that many preservice teachers lacked a comprehensive understanding of when and how to apply specific proof methods to different mathematical scenarios. Some participants indicated a tendency to use direct proof for “straightforward” statements and to switch to indirect methods (contraposition or contradiction) when direct proof did not seem feasible. This approach suggests a partial understanding of proof method applicability, as exemplified in the following statements:

1. **S003**: "For question 1, I used direct proof because it is straightforward."
2. **S004**: "I usually begin by analyzing the statement and check which proof method is suitable by reminding myself of their properties."
3. **S005**: "For straightforward ones, I use direct proof, but I know that I may face challenges at times. So, when I face challenges, for example in question 2 and question 3, I take the opposite, such as contrapositive or contradiction."
4. **S008**: "For item 1, I decided to use the direct proof method because it is not very challenging to me, and it only requires general knowledge of what odd numbers and even numbers are."

These responses align with the findings from the written scripts, where some participants exhibited a preference for familiar methods, such as direct proof, for simpler tasks but struggled when deeper reasoning or alternative proof methods were required. This reliance on surface-level understanding, as indicated by S003 and S008, coupled with uncertainty in selecting the appropriate method, as observed in S005, points to an inadequate understanding of proof concepts, further corroborating the partially correct or incorrect solutions noted in the written scripts. Nonetheless, it is important to recognize that preservice teachers may find some proofs more intuitive, which could lead to natural exploration of several methods before arriving at the correct one.

Regarding the challenges faced during the process, the following direct quotes illustrate the experiences of the preservice teachers:

1. **S001:** It was difficult for me to give conclusion before even proving. Proving made me uncertain, especially that I do not even believe in assumptions, but I did anyway. I know that n and m is odd

right! By definition, I first said $n = 2r$ (assumption) then I proved till I saw that if I add 1 to $2r$ then I will get an odd integer.
2. **S004:** Choosing the method to work with is kind of confusing sometimes and not knowing how to agree (direct proof) or disagree with the statement to prove a thing is a challenge to me.
3. **S005:** For question 3, I always get confused to understand why if 3 is a factor of a^2 then it should also be a factor of a. Because of language, I always confuse terminologies like factors and multiples. For example, in class we were saying a^2 is a multiple of 3 and not a factor of 3.
4. **S006:** For question 2, I first wanted to prove it using the direct proof method, but I did not know how to start. Later I remembered that I needed to use another method, but I had forgotten how to do it.

These interview responses reveal several misconceptions that correspond with the patterns observed in the written test scripts. They suggest that this group of preservice teachers demonstrated limited flexibility in their reasoning. For example, discomfort with using assumptions (e.g., S001's reluctance to believe in assumptions) and difficulty in connecting assumptions to conclusions point to a limited understanding of the logical structure of proofs. Additionally, the inability to initiate a proof, as seen in S006, reflects an absence of initial heuristic strategies.

Despite these challenges, some interviewees demonstrated a solid understanding of proof methods and their application in the given tasks. Examples from this group include:

1. **S008:** I did not face any difficulties while constructing a proof because I have a deep understanding of terms like odd numbers and how they operate.
2. **S003:** I had no difficulties
3. **S009:** I never had challenges because I was always sure of how to proceed. I have understood the logic behind these proofs.
4. **S002:** In using proof by contradiction to prove that $\sqrt{3}$ is irrational, the difficulties I had were from the misconception of negation. I was trying to work toward proving the statement as true instead of false. That means I lost track of the assumption I was working on.

Regarding whether they relied on memorization or a thorough understanding of proofs, four of the interviewed preservice teachers stated that they understood the underlying logic, while the remaining five indicated that they relied on memorization of the steps. When asked what they considered key to understanding proof methods, all participants emphasized that "practice" was essential. This suggests that the participants viewed increased practice as vital to improving their understanding of proof construction.

The findings of this study offer significant insights into preservice teachers' engagement with proof construction, the challenges they faced, and the underlying reasoning processes involved. The preservice teachers exhibited varying levels of proficiency in articulating claims, presenting data, and justifying their reasoning. Direct proof construction was handled relatively well, with 62.5% of participants providing correct solutions for Item 1. This result is consistent with the findings of Demiray and Bostan (2015), who observed that preservice teachers tend to favor direct proof and mathematical induction due to their perceived simplicity. Similarly, Brown (2018) noted that students are more likely to engage with direct proofs, viewing them as more convincing and easier to apply. However, despite their positive engagement with direct proof tasks, the results revealed that some participants failed to specify critical details, such as explicitly defining integers in their representations, which compromised the logical precision of their proofs. This finding aligns with Doruk (2019), who reported that preservice teachers often struggle to articulate definitions and apply them correctly in proof construction.

The analysis using Toulmin's model of argumentation highlighted gaps in components such as warrants, backing, and rebuttals, particularly in Items 2 and 3, which involved the application of proof by contraposition or contradiction. For instance, in proving the irrationality of $\sqrt{3}$ (Item 3), only 30.6% (n = 22) of responses were correct, while 68.0% were partially correct. The partially correct responses were primarily due to the lack of explicit definitions, such as the term "coprime integers," and essential assumptions like $b \neq 0$. The absence of these critical details weakened the logical structure of the arguments, mirroring findings by Siswono et al. (2020) and Zeybek and Galindo (2014), who noted similar difficulties in indirect proofs and the application of definitions. Similarly, in Item 2, misconceptions related to contraposition and contradiction led to irrelevant algebraic manipulations and logical inconsistencies. These difficulties underscored the challenges preservice teachers faced in establishing clear warrants and backing to support their claims. Several participants also failed to address rebuttals or provide clear connections between assumptions and conclusions, reflecting an incomplete understanding of the logical structures underpinning these proof methods. Weber (2004) and Azrou and Khelladi (2019) also observed that students often lack metacognitive awareness about proofs, which impedes their ability to grasp the deeper principles that guide proof construction.

Moreover, the study revealed a predominant reliance on procedural approaches rather than conceptual understanding among preservice teachers. This reliance on mimicking procedures, rather than engaging with the underlying principles of proof structures and methods, is consistent with observations by Buchbinder and McCrone (2020) and Moore (1994), who found that students often reproduce proofs without fully understanding their theoretical foundations. Such tendencies raise significant concerns for mathematics education, as preservice teachers' limited comprehension of proofs may hinder their ability to foster mathematical reasoning and critical thinking in future classrooms.

To address these challenges, teacher education programs must emphasize not only procedural fluency but also a deeper conceptual understanding of proofs. Mukuka et al. (2019) argue that strengthening proof comprehension is essential to equipping future teachers with the skills necessary to guide students in constructing logical arguments and identifying errors. One promising approach to achieving this goal is to provide preservice teachers with opportunities to read and critically assess their own proofs, as suggested by Mejia-Ramos et al. (2012). Such reflective practices could offer valuable insights into preservice teachers' understanding of proofs and highlight the specific difficulties they encounter. These findings are critical for informing the design of targeted interventions aimed at enhancing proof comprehension among preservice teachers.

While this study provides valuable insights, the findings should be interpreted with caution, particularly with regard to the educational context in which the study was conducted. The study took place at a single South African university, with third-year preservice teachers enrolled in a teacher education program. Given that these students are being trained to teach mathematics rather than pursue advanced mathematical research, the proofs presented were designed to enhance their mathematical knowledge for teaching. Consequently, the scope and complexity of the proofs may differ from those encountered by preservice teachers in other programs or regions, especially those training to become pure mathematicians. Therefore, the results may not be fully generalizable to preservice teachers in vastly different educational settings, both within and outside South Africa.

To address these limitations and enhance the robustness of future research, it is crucial to expand the scope of similar studies to include multiple institutions. Comparative studies examining preservice teachers' proof comprehension across different educational systems and levels of teacher training (e.g., mathematics specialists versus general teacher education programs) could provide a broader understanding of the challenges and best practices in proof instruction. Additionally, future research could adopt a mixed-methods approach, combining quantitative data with qualitative findings to enhance generalizability. Finally, longitudinal studies tracking the development of proof comprehension from preservice training to classroom practice would offer deeper insights into how proof skills are acquired and applied in real-world teaching environments.

CONCLUSION

This study aimed to explore preservice mathematics teachers' understanding and construction of mathematical proofs, with a focus on their reasoning processes and application of various proof methods. Through the use of Toulmin's argumentation model, the research assessed how preservice teachers justify their reasoning in mathematical proofs, identified gaps in their arguments, and evaluated their engagement with the conceptual and logical foundations of proof construction. The findings revealed several critical deficiencies, particularly in components such as warrants, backing, and rebuttals. Additionally, challenges were evident in tasks involving proof by contraposition and proof by contradiction. While preservice teachers generally displayed procedural fluency in constructing direct proofs, their failure to incorporate precise definitions and logical rigor suggested a superficial understanding of proof concepts. This highlights the need for mathematics teacher education programs to emphasize the development of a deeper, more conceptual understanding of proofs, ensuring that preservice teachers are better prepared to teach proof construction effectively.

Despite the valuable insights gained from this study, there are several limitations to consider. Firstly, the sample size was limited to a specific group of preservice teachers, which may not fully represent the broader population of mathematics teacher candidates. The study also primarily focused on the analysis of proof tasks within a controlled environment, which may not fully capture the complexity of proof construction in real classroom settings. Furthermore, the research was constrained by the use of Toulmin's model alone, which, while effective in analyzing logical structure, may not have accounted for other influencing factors, such as prior knowledge or individual teaching styles. These limitations suggest that the results should be interpreted with caution, and further studies are necessary to explore the broader applicability of the findings.

Future research in this area should aim to address these limitations by expanding the scope of the study to include a larger and more diverse group of preservice teachers from different institutions and training programs. Additionally, employing a mixed-methods approach or longitudinal study design could provide deeper insights into how preservice teachers' proof skills develop over time and how they apply these skills in classroom settings. Exploring proof comprehension in various educational contexts and comparing the effectiveness of different pedagogical strategies for teaching proofs could also contribute to a more comprehensive understanding of how to enhance preservice teachers' mathematical reasoning and argumentation skills. These future directions would help inform the design of more effective teacher education programs that support the development of critical thinking and logical reasoning in mathematics education.

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Author Contribution	:	AM: Conceptualization, data curation, formal analysis, investigation, methodology, project administration, visualization, validation, writing – original draft, and writing - review & editing. BT: Data curation, formal analysis, funding acquisition, investigation, methodology, project administration, validation, visualization, and writing - review & editing.
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