Journal on Mathematics Education
Volume 16, No. 1, 2025, pp. 365-382

The 'mound-hollow' model for solving integer addition and subtraction problems

Puspita Sari[1,*], Jaguthsing Dindyal[2], Paul Maurice Edmund Shutler[2]

[1]Mathematics Education Study Program, Universitas Negeri Jakarta, Jakarta, Indonesia
[2]National Institute of Education, Nanyang Technological University, Singapore
*Correspondence: puspitasari@unj.ac.id

Received: 7 November 2024 | Revised: 17 March 2025 | Accepted: 20 March 2025 | Published Online: 23 March 2025

Abstract

Understanding integer operations is a fundamental yet challenging concept for elementary students, often requiring effective visual models to support their comprehension. Despite various instructional models, many students continue to struggle with integer addition and subtraction, particularly when negative numbers are involved. Addressing this gap, this study explores the potential of the mound-hollow model to facilitate students' understanding of integer addition and subtraction concepts intuitively. This study aimed to examine how three sixth-grade students utilized the mound-hollow model to solve integer addition and subtraction problems. Data were collected from students' written tests and individual interviews conducted after a teaching experiment involving 25 sixth graders in Indonesia. The findings indicate that the mound-hollow model provides a meaningful analogy for solving addition problems of types $x + (-y)$ and $(-x) + y$ (where $x > y$ and x, y are natural numbers) and subtraction problems of types $x - (-y)$ and $(-x) - y$. All three students successfully employed the model to solve the addition problems by neutralizing every mound-hollow pair and to solve the subtraction problems by creating mound-hollow pairs. Additionally, students demonstrated the ability to justify their solutions and correct errors through the mound-hollow representation. The use of a single mound or hollow to represent larger integers enhanced students' proficiency in solving integer operations and reinforced their understanding of the relationship between addition and subtraction, such as $x - (-y) = x + y$ and $(-x) - y = (-x) + (-y)$. These findings highlight the effectiveness of the mound-hollow model as an alternative instructional tool for teaching integer operations, providing students with an intuitive framework to construct abstract mathematical concepts. The implications of this study contribute to mathematics education by offering insights into the design of visual models that support conceptual understanding in integer arithmetic.

Keywords: Addition, Integers, Mound-Hollow Model, Negative, Subtraction

How to Cite: Sari, P., Dindyal, J., & Shutler, P. M. E. (2025). The 'mound-hollow' model for solving integer addition and subtraction problems. *Journal on Mathematics Education, 16*(1), 365–382. https://doi.org/10.22342/jme.v16i1.pp365-382

Research has shown that many students encounter significant challenges when solving problems involving integer addition and subtraction. These difficulties can be attributed to three primary factors. First, students often struggle with counterintuitive situations that contradict their routine understanding of whole number operations, such as the misconceptions that addition always results in a larger value and subtraction always reduces a quantity (Bishop et al., 2014). Second, many students fail to distinguish between the different roles of the minus sign, namely: as an operator for subtraction (binary), as an indicator of a negative number (unary), and as a symbol for taking the opposite (symmetric) (Bofferding, 2014; 2019; Gallardo & Rojano, 1994; Vlassis, 2004; 2008). Consequently, students frequently disregard the negative sign in expressions like (-4) and incorrectly compute 5 − (-4) as if (-4) were a positive integer. Third, the absence of concrete and meaningful representations that accurately convey the concept of negative integers contributes to students' misconceptions. This challenge arises from the common interpretation of negative integers as values less than zero or as representing "nothing" (Goldin & Shteingold, 2001; Vig et al., 2014). As a result, students often rely on procedural rules without developing a deep conceptual understanding, leading to errors in arithmetic operations. Moreover, difficulties in performing integer arithmetic have been recognized as a contributing factor to students' struggles in early algebra (Gallardo & Rojano, 1994; Vlassis, 2004; 2008).

Previous research has explored various instructional models and contextual approaches to facilitate students' understanding of integer addition and subtraction. Broadly, two primary instructional models have been widely employed in teaching these concepts: the number line model and the neutralization (or cancellation) model (Kilpatrick et al., 2001). However, some scholars categorize their models based on specific contexts or metaphors used in their studies, such as the 'charged particle' model (Battista, 1983), the 'hot-air balloon' model (Janvier, 1985), the 'integers microworld' model (Thompson & Dreyfus, 1988), the 'scores and forfeits' model (Liebeck, 1990), the 'vertical number line' model (Stephan & Akyuz, 2012), virtual manipulatives incorporating both the number line and chips model (Bolyard & Moyer-Packenham, 2012), the 'banking' model (Shutler, 2017), and the 'walk-it-off' model (Nurnberger-Haag, 2018). While these models have demonstrated effectiveness in teaching integer addition, researchers have noted persistent challenges in their applicability to integer subtraction, necessitating further investigation. For instance, Janvier (1985) acknowledged the limitations of the 'hot-air balloon' model in addressing subtraction problems involving negative integers, while Nurnberger-Haag (2018) highlighted difficulties in using the neutralization model for solving subtraction problems with large integers.

Several research indicates that students encounter greater difficulties in solving integer subtraction problems compared to addition problems (Kilhamn, 2018; Küchemann, 1981; Liebeck, 1990). The number line model has been deemed ineffective for teaching subtraction involving negative integers, as it necessitates a 'turn-around' movement that contradicts the intuitive notion of subtraction as 'taking away' (Küchemann, 1981). On the other hand, the neutralization model is said to have 'breaking points', at which the model is no longer useful in understanding more abstract concepts, such as, (-3) – 5, and 3 – (-5) (Vig et al., 2014). Students often struggle to determine whether to 'neutralize' (remove zero pairs) in addition problems or to 'add neutrals' (adding zero pairs) in subtraction problems, leading to confusion in problem-solving (Lytle, 1994). In such instances, Bofferding (2014) argued that the neutralization model may obscure the binary function of the subtraction sign, as subtraction problems inherently involve both addition and removal processes, potentially leading to conceptual misunderstandings.

Challenges in making sense of the integer subtraction concept are not only due to the lack of tangible and intuitive representations for negative integers, but also due to the gap between concrete representations and abstract mathematics. A rote manner of learning with manipulatives could cause students' failures in understanding concepts, thus, the abstraction process from concrete representations to formal mathematics is crucial in developing students' understanding (Clements & McMillen, 1996; Kamii et al., 2001). For example, Thompson and Dreyfus (1988) reported that in the context of a turtle walking on a number line, the rule on subtracting a negative subtrahend was found to be less intuitive and more troublesome. Consequently, students who used the number line model tended to rely on the given rules rather than deduce their own, that is, subtracting a negative subtrahend becomes adding the positive (the additive inverse), $x - (-y) = x + y$, as indicated in Liebeck (1990). Hence, the model is forgotten as soon as students acquire rules and procedures for solving integer problems, and they cannot utilize the model for their reasoning when they encounter difficulties.

This study introduces the mound-hollow model, a refined version of the neutralization model in which the additive inverse property plays a crucial role in students' reasoning. A prior investigation explored a model with similar characteristics (Sari et al., 2020), wherein positive and negative integers were represented as scores and forfeits and visually depicted as mounds and hollows, respectively. The findings revealed that this dual representation imposed significant cognitive demands on students, limiting its effectiveness in solving more complex integer problems. However, further analysis indicated that distinct visual representations of mounds and hollows supported students' understanding of positive and negative integers as additive inverses, thereby reinforcing their comprehension of the additive inverse property. These insights suggest that incorporating a more intuitive context is essential, allowing students to meaningfully relate to the model when encountering challenging integer operations.

A comparable model, known as the hills and dales model, is incorporated into the mathematics curriculum of New Zealand (New Zealand Government, 2014). However, the present study was conducted independently and is not associated with the New Zealand curriculum. The primary distinction between the hills and dales model and the mound-hollow model lies in the approach to introducing subtraction involving a negative subtrahend. The hills and dales model conceptualizes subtraction with a negative subtrahend as the removal of dales, which is interpreted as the creation of hills—essentially reinforcing the principle that subtracting a negative number is equivalent to adding its additive inverse, such as (-3) − (-2) = (-3) + 2. In contrast, the mound-hollow model does not introduce this ‘minus minus plus’ rule in the first place, students would have to identify the sign of integers both in the minuend and in the subtrahend prior to solving a subtraction problem, as solving (-3) – (-2) or (-2) – (-3) would mean differently. This approach allows students to develop a conceptual understanding of integer subtraction and to intuitively deduce the rule through meaningful analogies rather than relying on procedural memorization.

Integer Addition and Subtraction

A negative integer can be defined as a symbol of the form (-x) where the relation x + (-x) = 0 holds, for every natural number x. Although the concept of negative integer is defined symbolically, the introduction of the concept to students generally starts from familiar contexts, such as temperatures below zero, losses or debts, depths, opposite directions, etc. The ‘Experiences in Mathematical Discovery’ textbook (NCTM, 1970) assigned different meanings to positive and negative integers, i.e., 1) as gains and losses, and 2) as points on a number line. Hence, integers can be conceptualized as representing quantities as in the neutralization model, or as representing positions relative to zero as in the number line model. In the case of the number line model, the ‘School Mathematics Project’ textbook (Bolt et al., 1965) introduced negative integers as: 1) a change in position or shift; or 2) a position on the number line relative to the origin. Other studies, such as Thompson and Dreyfus (1988) and Ulrich (2012) interpreted integers as directed magnitudes or directed distances on the number line.

The arithmetic operation of addition can be explained as an aggregation (finding the total of two quantities), or as an augmentation (finding an increased value), while the operation of subtraction can be interpreted as a partitioning (“how many are left?”), a reduction (“what is the reduced price?”), a comparison (“what is the difference?”), or an inverse-of-addition (“how many more needed?”) (Haylock & Cockburn, 1997). When introducing concepts of addition and subtraction involving negative integers, addition is no longer associated merely with finding an increased value, or subtraction simply as finding a difference. Different interpretations of integers may lead to different conceptualizations of addition and subtraction involving negative integers. For example, when integers are conceptualized as representing quantities, addition is defined as finding a total amount of two quantities, and subtraction as finding how many lefts, where the notions of neutralization, (-x) + x = 0, and creating from nothing, 0 = (-x) + x, are two important properties of these operations (Freudenthal, 1983). On the other hand, when integers are conceptualized as points on a number line, adding means counting forward from one of the addends, and subtracting means counting backward from the minuend, including the rules of facing in the negative direction when negative integers involve.

Understanding the concepts of addition and subtraction involving negative integers necessitates a conceptual change, as these concepts are often incompatible with students' prior knowledge acquired through everyday experiences (Bofferding, 2014; 2019; Merenluoto & Lehtinen, 2004; Vosniadou & Verschaffel, 2004). Bofferding (2019) emphasizes that the transition from whole number understanding to negative number comprehension is inherently complex. Students without prior instruction on negative integers often disregard the negative sign, misinterpret it as a subtraction operator, or even treat it as zero when solving addition and subtraction problems involving negative numbers (Bofferding, 2014; 2019). Given the intricacies of this conceptual shift, Bofferding (2019) highlights the critical role of context and language in facilitating a meaningful understanding of integer operations. In this regard, the mound-hollow model provides an intuitive framework that helps students grasp the fundamental principle that the sum of an integer and its additive inverse is zero, e.g., 1 + (-1) = 0. This understanding challenges the initial belief that "addition cannot make smaller," reinforcing the necessity of a conceptual change in students' reasoning.

Wessman-Enzinger and Mooney (2014; 2019) proposed four conceptual models—bookkeeping, counterbalance, translation, and relativity—for reasoning about integer addition and subtraction, which serve as a foundation for developing instructional models. Bofferding (2019) organized the four conceptual models in relation to contexts and instructional models used in teaching integer operations. For instance, the asset and debt context introduced by Stephan and Akyuz (2012) aligns with the bookkeeping model, as it allows students to conceptualize positive and negative integers as assets and debts, with zero representing a state of having neither. Similarly, the colored-chip model, used to represent scores and forfeits (Liebeck, 1990), corresponds to the counterbalance model, where students perceive negative values (forfeits) and positive values (scores) as two separate quantities that neutralize each other.

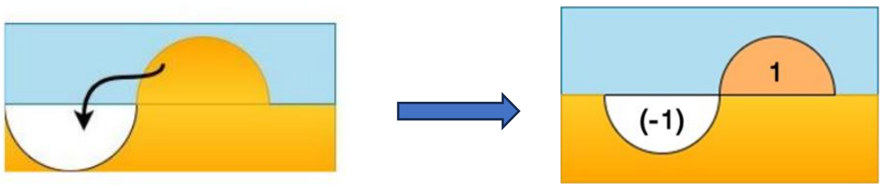
Bofferding (2019) further identified the directed movements context (Thompson & Dreyfus, 1988) as an example of an instructional model aligned with the translation conceptual model. In this approach, students conceptualize integers as vectors moving right or left on a number line, with zero functioning as a zero vector. In contrast, the relativity conceptual model, as described by Wessman-Enzinger and Mooney (2014; 2019), is inherently more abstract and challenging than the other three models. This is because it requires students to interpret integers in relation to arbitrary reference points, where zero can be defined differently depending on the context. However, real-world contexts such as temperature and elevation can support students' reasoning within the relativity conceptual model by providing tangible examples of how numbers represent relative positions or changes.

Given the importance of understanding students' conceptual frameworks before developing instructional models, Wessman-Enzinger and Mooney (2014; 2019) emphasized the necessity of examining students' reasoning about integer operations. In response, this study explores how the mound-hollow model facilitates students' understanding in terms of the counterbalance conceptual model, wherein mounds (positive integers) and hollows (negative integers) naturally neutralize each other. The mound-hollow model provides an intuitive foundation for grasping the principle of additive inverses, a fundamental concept in integer addition within the counterbalance framework. By engaging with this model, students can develop a concrete understanding of how an integer and its additive inverse sum to zero, thereby reinforcing their reasoning about integer operations in a meaningful and structured manner.

The Instructional Design Featuring the Mound-Hollow Model

The mound-hollow model is fundamentally the neutralization model, in which integers are conceptualized as numbers denoting quantities, while addition and subtraction are interpreted as the processes of adding and taking away, respectively. Figure 1 presents a two-dimensional depiction of mounds and hollows as key elements of this model, where each unit of mound and hollow is assumed to be congruent. In this framework, a negative integer is represented as a quantity of hollows, while a positive integer corresponds to a quantity of mounds. Simultaneously, negative integers are also understood as the additive inverses of positive integers—analogous to the idea that a hollow and a mound neutralize each other when combined.

To mitigate potential confusion between the minus sign used for subtraction and the negative sign denoting a negative integer, this study adopts a notation system in which negative integers are enclosed within brackets, (-x). This notation aims to provide clarity and consistency, ensuring that students can accurately distinguish between operations and integer signs when solving arithmetic problems involving negative numbers.



The quantity of mounds as a positive integer **1 + (-1) = 0**

The quantity of hollows as a negative integer

Figure 1. The mound-hollow representation

The common neutralization model typically uses chips or counters of two different colours or signs to represent positive and negative integers, in which the neutralization principle: x + (-x) = 0 is not self-evident. For example, red chips with minus sign and white chips with plus sign (Battista, 1983), black and red counters (Liebeck, 1990), black and white chips (Lytle, 1994), blue and yellow beads on a double abacus (Linchevski & Williams, 1999), and virtual integer chips (red and black) applet (Bolyard & Moyer-Packenham, 2012), Unlike these models, features of the mound-hollow model allow students to intuitively abstract the concept of neutralization: x + (-x) = 0 by using the ‘landslide’ analogy, i.e., a mound which slides into a hollow of the same magnitude will become flat ground (flat ground is interpreted as nothing or zero). In Bahasa Indonesia, “landslide” (noun) or “slides” (verb) translates to *longsor,* which is a common occurrence in Indonesia, due to the steep topography and high annual rainfall. The analogy of *longsor* originally came from one of the students during the teaching intervention when they were combining a mound and a hollow together and see that a mound and a hollow could turn into flat ground by exploring the concrete manipulative. This proved that Indonesian students are quite familiar with this analogy.

The mound-hollow representation clearly distinguishes positive and negative integers, so different colors or signs are no longer needed. The image of a zero pair, e.g., 1 and (-1), can be represented as a mound-hollow pair, and not as two identical chips with distinct colors or signs as in the common neutralization model. This image of mound-hollow pairs (zero pairs) is intended to facilitate students' reasoning while solving difficult subtraction problems between a positive and a negative integer, such as 6 – (-2) or (-6) – 2. The act of creating mound-hollow pairs to represent adding zero pairs in some types of subtraction problems is called the 'digging' strategy. Thus, students could intuitively use the analogy of *gali* (in Bahasa Indonesia), which means "dig" or "excavate" to explain the act of digging a hollow and thus creating a mound of the same magnitude.

In this study, we identified 16 possible types of integer addition and subtraction problems based on the magnitude, the order and the sign of the integers, where x and y are natural numbers with x ≥ y:

Eight types of integer addition			Eight types of integer subtraction		
x + y	and	y + x	x – y	and	y – x
x + (-y)	and	(-y) + x	x – (-y)	and	(-y) – x
(-x) + y	and	y + (-x)	(-x) – y	and	y – (-x)
(-x) + (-y)	and	(-y) + (-x)	(-x) – (-y)	and	(-y) – (-x)

We subsequently refined these 16 types into ten distinct categories, comprising four types of integer addition problems (A1 to A4) and six types of integer subtraction problems (S1 to S6) as summarized in Table 1.

Table 1. Ten types of integer addition and subtraction problems

Type	Code	Example	Note
x + y or y + x	A1	3 + 7	Addition of two positive integers OR two negative integers.
(-*x*) + (-*y*) or (-*y*) + (-*x*)	A2	(-3) + (-7)	
x* + (-*y*) *or* (-*y*) + *x	A3	7 + (-3) or (-3) + 7	Addition of a positive and a negative when the magnitude of the positive is larger than the negative.
(-*x*) + *y* or *y* + (-*x*)	A4	(-7) + 3 or 3 + (-7)	Addition of a positive and a negative when the magnitude of the negative is larger than the positive.
x – y	S1	7 – 3	Subtraction between two positive OR two negative integers, when the magnitude of the minuend is larger than the subtrahend.
(-*x*) – (-*y*)	S2	(-7) – (-3)	
y – x	S3	3 – 7	Subtraction between two positive OR two negative integers, when the magnitude of the minuend is smaller than the subtrahend.
(-*y*) – (-*x*)	S4	(-3) – (-7)	
***x* – (-*y*) or *y* – (-*x*)**	S5	7 – (-3)	Subtraction between a positive integer and a negative integer.
(-*x*) – *y* or (-*y*) – *x*	S6	(-7) – 3	

This study presents specific findings from a design-based research (DBR) study conducted in a sixth-grade classroom in Indonesia, involving 25 students (Sari, 2023). The research focused on implementing designed instruction using the mound-hollow model to facilitate students' understanding of integer addition and subtraction concepts. The discussion is limited to four specific problem types—A3, A4, S5, and S6—as solving these problems requires a conceptual change, particularly in understanding that addition can result in a smaller value and subtraction can result in a larger value.

Moreover, addition problems of types A3 and A4 necessitate the ‘landslide’ analogy, while subtraction problems of types S5 and S6 are known to be particularly challenging for students (Küchemann, 1981; Peled et al., 1989). Therefore, this study aims to address the following research questions:

1. How do students utilize the ‘landslide’ analogy when solving integer addition problems of the forms x + (-y) and (-x) + y for x > y?
2. How do students apply the ‘digging’ strategy in solving integer subtraction problems of the forms x – (-y) and (-x) – y?

METHODS

Participants

Prior to the instructional intervention, a written pre-test (Table 2) was administered to all 25 sixth-grade students in the class, followed by individual interviews with twelve selected participants. The aim of the pre-test was to understand students’ prior knowledge about negative integers and the operation of addition and subtraction involving negative integers. The duration of the written pre-test was half an hour, and the individual interviews were 15 minutes per student on average. Although none of the seven questions in the pre-test showed negative integers, the solutions did require students’ understanding of the negative integer concept. While solving the questions, students were encouraged to write down their reasoning, even if they thought that the questions could not be solved. The three students (NP1, NP2, NP3) were selected because they were indicated as students who had no prior knowledge, and they showed active participation in all five lessons during the intervention. Based on pre-test results, interview, and the classroom teacher’s information, NP1, NP2, NP3 were categorized as high, average, and low performing students, respectively. Participation in this study was entirely voluntary, and all identifying information of the participants has been anonymized in this paper.

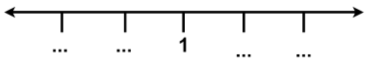
Data Collection Methods

The data analyzed in this study were derived from students’ written post-tests and individual interviews conducted following the instructional intervention, as well as from students’ worksheets and field notes collected during the teaching intervention. The analysis employed the constant comparative method, in which each unit of data was systematically examined and compared with previously analyzed data (Mathison, 2005). Through this process, four key terms associated with students’ reasoning in relation to the mound-hollow model were identified: mounds, hollows, landslide, and digging. Consequently, while analyzing students’ written test responses and interview transcripts, comparisons were made with previously analyzed data gathered during the instructional intervention.

In this study, the post-test was not intended to measure students’ improvement in solving integer tasks, but to inform our analysis and to deepen our understanding about students’ development post intervention. Of the ten questions in the post-test, this paper only focusses on four of them, i.e., addition of a positive and a negative integer: (-7) + 21, and 9 + (-100), and subtraction between a positive and a negative integer: (-9) – 15 and 1 – (-1000),which represent the four problem types (A3, A4, S5, and S6) mentioned earlier. Although the focus was only on four of the post-test questions, other data collected during the teaching intervention was also reported in this paper.

The individual interview after the written post-test was intended to clarify students’ particular answers and to seek deeper understanding of students’ thinking and reasoning process. The researcher usually started the interview session by asking general questions, such as: “How did you solve it?”, “Can you explain your solutions (or drawings)?" or "How did you perform this addition (or subtraction)?" More specific questions and instructions were given as a follow up of students' responses indicating the use of the mound-hollow model. During the interview, some students were given additional questions, for example, to check the correctness of: '(-30) – 50 = (-20)' and '1 – (-100) = 1 + 100'.

Table 2. Pre- and post-test questions

Pre-Test Questions	Post-Test Questions
1. 42 – 38 = …	1. (-70) + (-13) = …
2. … + 7 = 5	2. (-7) + 21 = …
3. 10 + … = 0	3. 9 + (-100) = …
4. 4 – … = 10	4. 4 – … = 10
5. 2 – 5 = …	5. (-42) – (-38) = …
6. 0 – 5 = …	6. (-5) – (-9) = …
7. Fill in the blanks with the correct numbers 	7. (-9) – 15 = …
	8. 0 – 500 = …
	9. 1 – (-100) = …
	10. (-8) + 10 – 15 = …

RESULT AND DISCUSSION

This section describes our analysis of students' performances in utilizing the mound-hollow model for solving integer addition and subtraction problems, specifically, the 'landslide' analogy for solving addition problems: $x + (-y)$ and $(-x) + y$, for $x > y$, where x and y are natural numbers, and the 'digging' strategy for solving subtraction problems: $x - (-y)$ and $(-x) - y$.

The 'Landslide' Analogy for Solving Integer Addition: x + (-y) and (-x) + y, x > y

In adding positive and negative integers, students can combine a group of mounds and a group of hollows followed by applying the 'landslide' analogy, $x + (-x) = 0$. Adding two integers of the same sign, however, does not require the neutralization principle.

During the intervention, students were consistently encouraged to articulate their reasoning, with the flexibility to either illustrate their solutions using mounds and hollows or express them verbally. In the first few lessons, students represented integers by drawing each mound or hollow separately one by one, leading to a preference for verbal explanations when dealing with problems involving large integers. As the complexity of integer problems increased, particularly those involving larger magnitudes, students encountered challenges in visually representing excessive numbers of mounds and hollows. Those with no prior knowledge of solving such problems made considerable efforts to illustrate as many mounds and hollows as necessary. However, one student introduced an innovative approach by using a single mound or hollow to symbolize 10 or (-10), respectively. This concept was subsequently discussed in class, leading students to adopt the practice of using a single mound or hollow to represent groups of ten, which later evolved into a generalized method for representing any integer.

The post-test result presented in Figure 2 shows that there are similar patterns from the three students' solutions of the two addition problems, (-7) + 21 and 9 + (-100). For example, Figure 2(i) illustrates how NP1 and NP2 carefully drew mounds (hollows) one by one representing ones, while NP3 used a single mound and hollow to represent 21 and (-7). The curved arrows in students' drawings indicated the 'landslide' analogy, where an equal number of mounds and hollows cancels each other out.

Although NP3 initially had written (-14) as a solution to (-7) + 21, he himself was able to correct the mistake. He could refer back to the mound-hollow context and determined the correct sign of the sum by perceiving the remaining mounds or hollows.

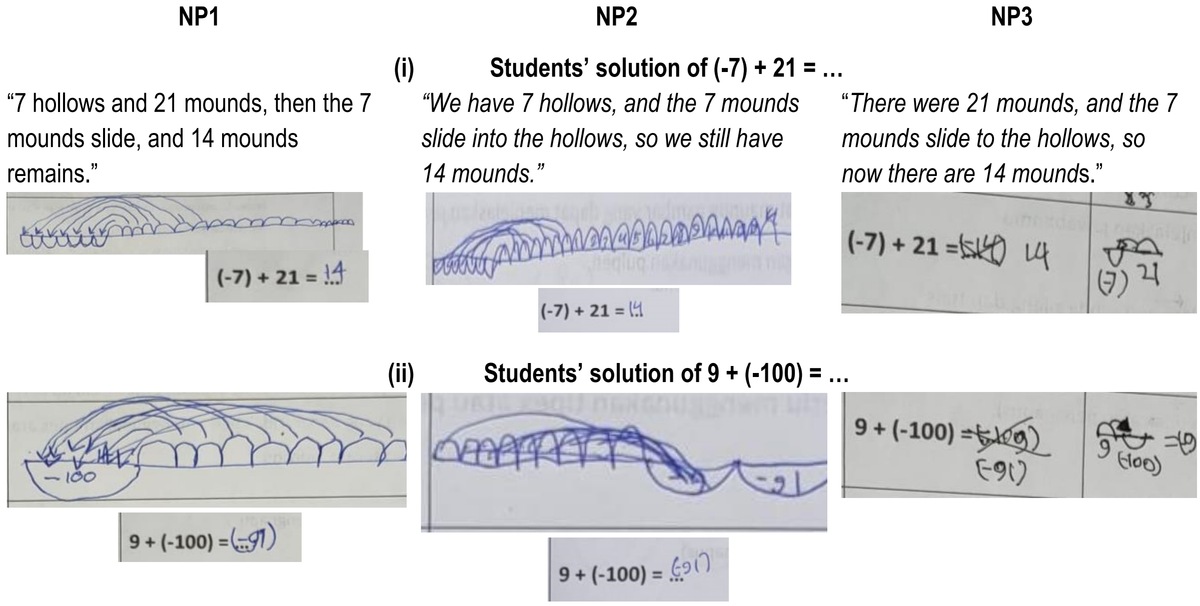


Figure 2. Students' solution of (-7) + 21 and 9 + (-100)

NP1 and NP2 started to use a single hollow for problems involving large integers (Figure 2(ii)). NP2's representation of (-100) indicates that NP2 probably had found the difference between magnitudes, 100 and 9, to split (-100) as two hollows of (-9) and (-91). This shows that NP2 has a good understanding of addition of two negative integers, i.e. (-100) = (-9) + (-91). On the other hand, it seemed that NP3 had initially answered (-109) for 9 + (-100), but his drawing shows that he performed the 'landslide' analogy in finding the correct answer, that is (-91).

The 'landslide' analogy proved to be an effective tool for all three students in meaningfully solving addition problems involving positive and negative integers, even when dealing with large numerical values. Notably, NP3, identified as a low-performing student, was able to utilize the mound-hollow model to reflect on his initial response and subsequently correct procedural errors. Through the application of this analogy, students established connections between integer problems and the mound-hollow context, enabling them to articulate their problem-solving strategies using the model. Over time, they developed an understanding that adding a positive and a negative integer is conceptually equivalent to subtracting the smaller magnitude from the larger one and assigning a negative sign to the result when the negative integer has a greater magnitude. This reasoning aligns with commonly observed strategies for determining the sum of a positive and a negative integer, as documented in previous studies (Kilhamn, 2018; Wessman-Enzinger, 2019; Aqazade & Bofferding, 2021; Bofferding et al., 2017).

The process of canceling out equal numbers of mounds and hollows in the 'landslide' analogy corresponds to the counterbalance conceptual model proposed by Wessman-Enzinger and Mooney (2014; 2019). Various instructional models incorporating different real-world contexts have been explored to promote this counterbalance conceptual model. Examples include the "happy and sad thoughts" model, where happy (positive) thoughts counterbalance sad (negative) thoughts (Whitacre et al., 2012); the "scores and forfeits" model, in which a score (positive) cancels out a forfeit (negative) (Liebeck, 1990);

the "assets and debts" model, where assets counterbalance debts (Stephan & Akyuz, 2012); and the red-and-white chip model, where red chips represent negative values and white chips represent positive values, with pairs canceling each other out (Battista, 1983). Compared to these models, the mound-hollow model with the 'landslide' analogy, provides a more intuitive context and distinct visual representation of positive and negative integers as additive inverses. This distinct visual representation also facilitated students' ability to implement the 'digging' strategy, which proved particularly useful for solving complex subtraction problems, as will be elaborated in the subsequent section.

The 'Digging' Strategy for Solving Integer Subtraction: x – (-y) and (-x) – y

In this study, the concept of subtraction as taking away or removing was introduced very carefully, because there could be potential difficulties when students were subtracting a negative subtrahend. Solving a subtraction problem of type x – y, where x > y can be interpreted as removing concrete objects that represent mounds, that is, removing y number of mounds from the available x mounds. However, unlike mounds that can be removed physically, hollows are non-tangible objects that cannot be removed physically, as a hollow is interpreted as an absence of soil in the ground. This study anticipated students' difficulties while dealing with subtraction problems with a negative subtrahend, such as, x – (-y) or (-x) – (-y), by focusing students' attention on the drawing representation of mounds and hollows, instead of mounds and hollows in the real-world context. Therefore, to subtract means to cross out the drawing representation of mounds (hollows) indicated in the subtrahend from the existing mounds (hollows) indicated in the minuend.

Prior to the discussion of problems of types x – (-y) and (-x) – y, students had solved the other four problem types: x – y; (-x) – (-y); y – x and (-y) – (-x), for x > y, by using mound-hollow model. The notion of creating mound-hollow pairs (zero pairs) from flat ground had emerged from previous classroom discussion as the teacher posed question: "*Can you create mounds and hollows from flat ground*?". The use of concrete manipulative which was designed to represent mounds and hollows helped students to make sense of the digging' strategy. Compared to the common neutralization model in which a pair of chips of two different colors or signs is used to represent adding zero pairs (e.g., in the studies of Liebeck (1990) and Lytle (1994)), a mound-hollow pair is concretely created from the ground in the mound-hollow context. 'Thus, to solve problems x – (-y) and (-x) – y, students could create (draw) zero pairs so that mounds (hollows) indicated in the subtrahend could be crossed out. Although the concrete manipulative was used in the beginning of intervention, students were encouraged to abstract the idea of 'digging' by drawing zero pairs.

Figure 3 presents the solutions of NP1 and NP2 to the subtraction problem (-9) – 15, utilizing the mound-hollow model. In this model, the problem (-9) – 15 is conceptualized as the removal of 15 mounds from 9 hollows. Since there were initially only 9 hollows and no mounds, students were required to create 15 zero pairs to facilitate the subtraction process. Both NP1 and NP2 successfully recognized (-9) – 15 as the removal of positive 15 (represented by 15 mounds) from negative 9 (represented by 9 hollows), thereby necessitating the creation of 15 zero pairs to enable the subtraction operation.

NP1 approached the problem systematically by first drawing 9 individual hollows, then adding 15 zero pairs before eliminating the 15 mounds, ultimately arriving at a final result of 24 hollows. Similarly, NP2 demonstrated the ability to translate the integer subtraction problem into the mound-hollow model, effectively applying the 'digging' strategy to solve it. NP2's illustration suggests that due to space limitations on the answer sheet, he opted to represent positive 15 with a single, larger mound instead of drawing 15 individual mounds, signifying the process of digging 15 hollows sequentially. Both NP1 and

NP2 continued employing the ‘digging’ strategy in subsequent problems, such as solving 1 – (-1000), where they used a single mound and a single hollow to represent 1000 and (-1000), respectively.

NP1 **NP2**

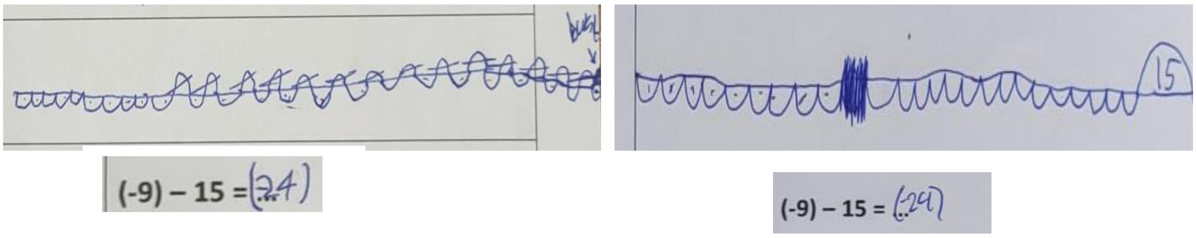


Figure 3. The digging strategy for solving subtraction (-9) – 15

Figure 4 illustrates NP3’s revision of his initial responses. In his initial attempt at solving the subtraction problem (-9) – 15, NP3 erroneously applied addition, combining (-9) and 15 rather than correctly subtracting 15 from (-9). The interview findings suggest that while NP3 was able to conceptualize negative and positive integers as representations of hollows and mounds, respectively, he overlooked the minus sign as the subtraction operator.

During the post-test interview, NP3 was asked to re-evaluate his solutions to the subtraction problems (-9) – 15 and 1 – (-1000). He successfully resolved these problems upon applying the ‘digging’ strategy. For instance, in solving (-9) – 15, NP3 confidently depicted a single hollow to represent (-9) and subsequently created a mound-hollow pair, consisting of 15 and (-15), to facilitate the removal of positive 15. He employed a plus sign to denote the addition of (-9) and (-15), thereby correctly arriving at the solution to (-9) – 15. The transcript presented in the following section is a translated version from Bahasa Indonesia, as the interview was originally conducted in that language.

Interviewer	:	*Now let’s draw the solution!*
NP3	:	*(Drawing a hollow and writing (-9) below it)*
Interviewer	:	*Ok, then, what do you want to subtract from?*
NP3	:	*Fifteen mounds*
Interviewer	:	*So, this is flat ground, do you have any mound here?*
NP3	:	*No, then I must dig. (Drawing a mound-hollow pair and writing (-15) and 15)*
Interviewer	:	*Ok, good. Then, what do you want to cross out?*
NP3	:	*This one, minus… (pointing to the hollow or negative 15)*
Interviewer	:	*Can you look at the number here in the problem?*
NP3	:	*Ups, I need to cross out this neg... (wanted to say ‘negative’, but unsure), Ups no, this one! (pointing to the mound which represent 15)*
Interviewer	:	*Why do you want to remove that one?*
NP3	:	*Because the problem asked me to remove the positive.*
Interviewer	:	*Ok and then...?*
NP3	:	*Then, I add (writing a plus sign in between the hollows (-9) and (-15))*

Utilizing the mound-hollow model by applying the ‘digging’ strategy simplified the cognitive process of simultaneously adding and removing values. This approach allowed students to bypass the need for explicitly incorporating zero pairs in mathematical notation, such as in the expression: 3 – (-5) = 3 + [5 + (-5)] – (-5). However, some students encountered difficulties, such as incorrectly canceling out the wrong mound or hollow after forming a mound-hollow pair.

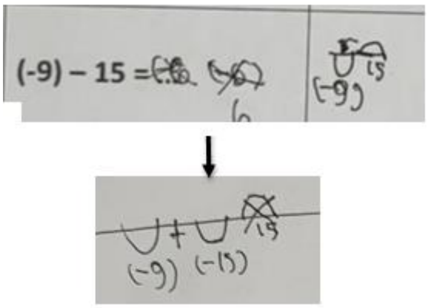


Figure 4. The digging strategy for correcting errors

Additionally, certain students struggled with interpreting subtraction as the removal of the subtrahend from the minuend. Instead, they erroneously treated both numbers as addends in an integer addition operation. This study identified several instances where students mistakenly performed addition rather than subtraction when solving expressions of the form x – (-y) and (-x) – y. Such errors, arising from the overgeneralization of addition in subtraction problems, have been previously documented in the literature (Ball, 1993; Bolyard & Moyer-Packenham, 2012; Küchemann, 1981; Schwarz et al., 1994). Nevertheless, the findings of this study indicate that at the end of the intervention, students were not only able to correctly solve subtraction problems of types S5 and S6, but they were also able to deduce relationships between subtraction and addition, which will be further elaborated in the subsequent section.

Making Sense of the Rules: x – (-y) = x + y and (-x) – y = (-x) + (-y)

Two additional subtraction problems of types (-*x*) – *y* and *x* – (-*y*) were given to examine whether students could utilize the 'digging' strategy in developing rules for solving the two types of subtraction problems. NP1 was asked to check the correctness of '(-30) – 50 = (-20)' and 1 – (-177) = 1 + 177 (Figure 5), and NP2 was asked to check the correctness of '(-30) – 50 = (-20)' and 1 – (-99) = 1 + 99 (Figure 6).

Figure 5 shows how NP1 solved the two problems. In solving (-30) – 50, NP1 initially drew a hollow (representing (-30)) and a mound (representing 50) together as if she had both (-30) and 50 (Figure 6(i)). However, she realized immediately that she could not draw the mound which represents 50, as it was a subtraction problem. Next, she proved that (-30) – 50 = (-20) was incorrect and the correct result was (-80) by utilizing the 'digging' strategy. This finding demonstrates how representing thinking with the mound-hollow model allowed her to recognize the error and to identify the problem as subtracting positive 50 from negative 30 and thus solve the problem correctly.

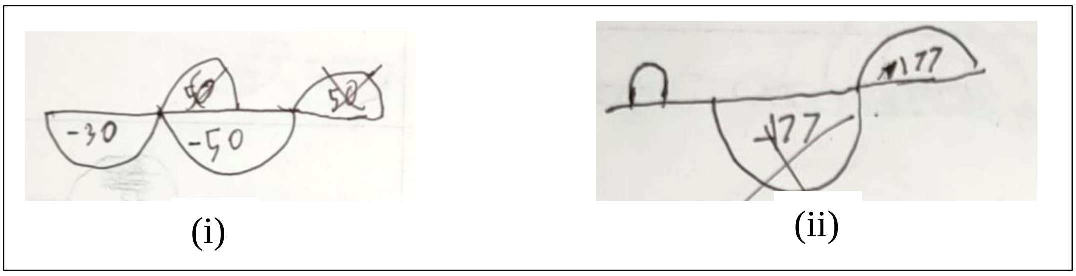
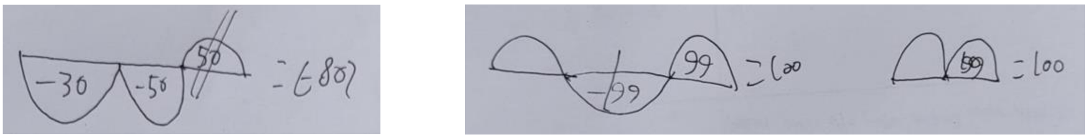


Figure 5. NP1's solutions to subtraction problems (i) (-30) – 50 and (ii) 1 – (-177)

By utilizing the ‘digging’ strategy, she could also show that 1 – (-177) = 1 + 177 by examining the results from both sides of the equation. Figure 5(ii) shows how she managed to solve 1 – (-177) correctly without any doubt and difficulty by using the mound-hollow model. The way she drew a mound-hollow pair as representing 177 and (-177) shows her improvement compared to Figure 2 when she drew smaller mounds and hollows representing ones. As a student who had no prior knowledge of the negative integer concept before the intervention, these findings demonstrate how she started to develop rules in solving subtraction problems by utilizing the ‘digging’ strategy in the mound-hollow model, such as, $x – (-y) = x + y$ and $(-x) – y = (-x) + (-y)$.



(i) (ii)

Figure 6. NP2’s solutions to subtraction problems (i) (-30) – 50 and (ii) 1 – (-99)

The interview with NP2 also confirmed that he benefited from the use of the ‘digging’ strategy in solving subtraction problems of types $(-x) – y$ and $x – (-y)$. Before the intervention, NP2 argued that problems like: … + 7 = 5 were impossible to solve, as “addition cannot make smaller”, and he treated subtraction problems $y – x$ as $x – y$ where $x > y$. After the intervention, NP2 managed to solve all post-test questions correctly, and he was consistent in utilizing the mound-hollow model while solving integer addition and subtraction problems. NP2 could associate the meaning of negative integers as quantities of hollows, positive integers as quantities of mounds, and the meaning of addition and subtraction operation as combining and taking away, respectively. Figure 6 and the interview transcript below show us that NP2 was able to show that ‘(-30) – 50 = (-20)’ was incorrect, by examining that the result of (-30) – 50 is (-80), using the ‘digging’ strategy in the mound-hollow model. He could also find out that 1 – (-99) is equal to 1 + 99 by examining the results from both sides. These findings show that students who had no prior knowledge about the negative integer concept could utilize the mound-hollow model, specifically, the ‘digging’ strategy in deducing rules $x – (-y) = x + y$ and $(-x) – y = (-x) + (-y)$.

Interviewer : So, if I have this problem, do you think (-30) – 50 is equal to (-20)?
NP2 : we have thirty …
Interviewer : You can write it down.
NP2 : Ok (start drawing a hollow and write (-30) in the hollow)
Interviewer : Is it thirty hollows?
NP2 : Yes, then we must take away fifty mounds. Because we don’t have mounds, so we need to dig. (drawing a mound-hollow pair and writing integers (-50) and 50)
Interviewer : Then …?
NP2 : Equals … (writing (-80) as the result)
Interviewer : So, is this incorrect?
NP2 : Yes, incorrect.
Interviewer : What if I have one take away negative ninety-nine, is it equal to one plus ninety-nine (while writing 1 – (-99) = 1 + 99))? Do you think this (pointing to 1 – (-99)) equals this (pointing to 1 + 99)?
NP2 : I think… Can I draw?

Interviewer : Sure, you can.
NP2 : One (drawing one mound), then we take away ninety-nine which we don't have. So… (drawing a mound-hollow pair, (-99) and 99), this one is crossed out (the negative 99). This is a hundred. (Figure 6 (ii)) This one (evaluating 1 + 99 by drawing one mound and another mound with 99 on it), same! The result is a hundred! (sounds surprise)

Despite subtraction problems of the forms x − (-y) and (-x) – y being recognized as among the most challenging in integer operations (Küchemann, 1981), students demonstrated their ability to apply the 'digging' strategy within the mound-hollow model to solve these problems effectively. The clear distinction between positive and negative integers as mounds and hollows, the intuitive nature of the 'digging' strategy for handling complex subtraction problems, and the flexibility of representing large integers using a single mound or hollow all contributed to students' reasoning processes in solving subtraction problems of these types. These findings address concerns regarding the limitations of the neutralization model (Vig et al., 2014) and the challenges associated with employing neutralization models that rely on chips of different colors or signs when dealing with subtraction involving large integers (Nurnberger-Haag, 2018).

This study provides empirical evidence that upper elementary school students can intuitively develop a conceptual understanding of integer addition and subtraction. The implications extend beyond mathematics educators to curriculum developers and textbook authors, highlighting the need to reconsider the predominant use of the number line model for introducing integer addition and subtraction operations in Indonesian curricula. Additionally, this research identifies ten distinct categories of integer addition and subtraction problems, offering a structured framework for teachers in designing instructional materials for these concepts.

CONCLUSION

This paper has demonstrated how three students effectively utilized the 'landslide' analogy to solve integer addition problems of types x + (-y) and (-x) + y when $x > y$, as well as the 'digging' strategy to solve integer subtraction problems of types x – (-y) and (-x) – y. Through a gradual process of abstraction, the students derived a systematic approach for solving these addition problems by subtracting the smaller absolute value from the larger one and assigning a negative sign to the result when the integer with the greater absolute value was negative. Additionally, they demonstrated an understanding of the fundamental relationship between addition and subtraction, particularly recognizing the equivalencies x – (-y) = x + y and (-x) – y = (-x) + (-y). The use of the mound-hollow model, specifically the 'digging' strategy, enabled students to justify their solutions and self-correct errors. Furthermore, their transition from representing small integers with unit mounds (or hollows) to using a single large-value mound (or hollow) for larger integers illustrated their progression in abstract reasoning. This shift not only enhanced students' proficiency in solving integer addition and subtraction problems but also reinforced their ability to establish conceptual connections between integer addition and subtraction operations.

Despite the effectiveness of the instructional approach in facilitating students' comprehension of integer operations, this study is limited by its small sample size, focusing on only three participants. As a result, the findings may not be broadly generalizable to a larger population of students with varying levels of mathematical proficiency. Additionally, the study primarily examined students' reasoning within a specific instructional framework, leaving room for further exploration of alternative models that may yield similar or improved outcomes. Future research could expand upon this work by investigating the application of the mound-hollow model in diverse classroom settings, incorporating a larger and more heterogeneous sample to assess its effectiveness across different learning contexts. Moreover, longitudinal studies could be conducted to examine the long-term retention of students’ conceptual understanding and problem-solving strategies in integer arithmetic. Exploring the integration of digital tools or interactive learning environments to enhance visualization and engagement in integer operations also presents a promising avenue for further research.

Declarations

Author Contribution	:	PS: Conceptualization, Methodology, Formal analysis, Writing - Original Draft, Editing, and Visualization. JD: Methodology, Analysis, Validation, Supervision, Writing – Review, and Editing. PMES: Methodology, Analysis, Validation, Supervision, Writing – Review, and Editing.
Funding Statement	:	The authors acknowledge funding from Universitas Negeri Jakarta.
Conflict of Interest	:	The authors declare no conflict of interest.
Additional Information	:	Additional information is available for this paper.

REFERENCES

Aqazade, M., & Bofferding, L. (2021). Second and fifth graders’ use of knowledge-pieces and knowledge-structures when solving integer addition problems. *Journal of Numerical Cognition*, *7*(2), 82-103. https://doi.org/10.5964/jnc.6563

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, *93*(4), 373-397. https://doi.org/10.1086/461730

Battista, M. T. (1983). A complete model for operations on integers. *Arithmetic Teacher*, *30*(9), 26-31. https://doi.org/10.5951/AT.30.9.0026

Bishop, J. P., Lamb, L. L., Philipp, R. A., Whitacre, I., Schappelle, B. P., & Lewis, M. L. (2014). Obstacles and affordances for integer reasoning: An analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education*, *45*(1), 19-61. https://doi.org/10.5951/jresematheduc.45.1.0019

Bofferding, L. (2014). Negative integer understanding: Characterizing first graders’ mental models. *Journal for Research in Mathematics Education*, *45*(2), 194-245. https://doi.org/10.5951/jresematheduc.45.2.0194

Bofferding (2019). Understanding negative numbers. In A. Norton & M. W. Alibali (Eds.), *Constructing number: Merging perspectives from psychology and mathematics education* (pp. 251-277). Springer International Publishing AG. https://doi.org/10.1007/978-3-030-00491-0_12

Bofferding, L., Aqazade, M., & Farmer, S. (2017). Second graders' integer addition understanding: Leveraging contrasting cases. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (pp. 243-250). Hoosier Association of Mathematics Teacher Educators.

Bolt, A. B., Holding, D. J., Tammadge, A. R., & Tyson, J. V. (1965). *The school mathematics project book 1* (A. G. Howson, Ed.). Cambridge University Press

Bolyard, J., & Moyer-Packenham, P. S. (2012). Making sense of integer arithmetic: The effect of using virtual manipulatives on students' representational fluency. *Journal of Computers in Mathematics and Science Teaching*, *31(2)*, 93-113. https://www.learntechlib.org/primary/p/39192/

Clements, D. H., & McMillen, S. (1996). Rethinking 'concrete' manipulatives. *Teaching Children Mathematics, 2*(5), 270-279. https://doi.org/10.5951/TCM.2.5.0270

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Springer. https://doi.org/10.1007/0-306-47235-x

Gallardo, A., & Rojano, T. (1994). School algebra syntactic difficulties in the operativity with negative numbers. In D. Kirshner (Ed.), *The Sixteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 159-165). Louisiana State University.

Goldin, G.A., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. A. Cuoco & F. R. Cuoco (Eds.), *The roles of representation in school mathematics (2001 yearbook)* (pp. 1-23). The National Council of Teachers of Mathematics, Inc.

Haylock, D., & Cockburn, A. (1997). *Understanding mathematics in the lower primary years* (Revised and expanded ed.). Paul Chapman Publishing Ltd.

Janvier, C. (1985). Comparison of models aimed at teaching signed integers. In L. Streefland (Ed.), *Proceedings of the Ninth Annual Conference of the International Group for the Psychology of Mathematics Education* Volume 1 (pp. 135-140). International Group for the Psychology of Mathematics Education.

Kamii, C., Lewis, B. A., & Kirkland, L. (2001). Manipulatives: When are they useful? *Journal of Mathematical Behavior, 20*(1), 21-31. https://doi.org/10.1016/S0732-3123(01)00059-1

Kilhamn, C. (2018). Different differences: Metaphorical interpretations of "difference" in integer addition and subtraction. In L. Bofferding & N. Wessman-Enzinger (Eds.), *Exploring the integer addition and subtraction landscape* (pp. 143-166). Springer. https://doi.org/10.1007/978-3-319-90692-8_6

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academy Press.

Küchemann, D. (1981). Positive and negative numbers. In K. M. Hart (Ed.), *Children's understanding of mathematics* Volume 11-16 (pp. 82-87). CSMS Mathematics Team.

Liebeck, P. (1990). Scores and forfeits: An intuitive model for integer arithmetic. *Educational Studies in Mathematics*, *21*(3), 221-239. https://doi.org/10.1007/BF00305091.

Linchevski, L., & Williams, J. (1999). Using intuition from everyday life in 'filling' the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics, 39*(1), 131-147. https://doi.org/10.1023/A:1003726317920

Lytle, P. A. (1994). Investigation of a model based on the neutralization of opposites to teach integer addition and subtraction. In J. P. d. Ponte & J. F. Matos (Eds.), *Proceedings of the 18th International Conference for the Psychology of Mathematics Education* Volume 3 (pp. 192-199).

Mathison, S. (2005). *Encyclopedia of evaluation*. Sage Publications. https://doi.org/10.4135/9781412950558

Merenluoto, K., & Lehtinen, E. (2004). Number concept and conceptual change: Towards a systemic model of the processes of change. *Learning and Instruction*, *14*(5), 519-534. https://doi.org/10.1016/j.learninstruc.2004.06.016

National Council of Teachers of Mathematics. (1970). Positive and negative numbers. In *Experiences in mathematical discovery.*

New Zealand Government. (2014). *Hills and dales.* https://nzmaths.co.nz/resource/hills-and-dales

Nurnberger-Haag, J. (2018). Take it away or walk the other way? Finding positive solutions for integer subtraction. In L. Bofferding & N. M. Wessman-Enzinger (Eds.), *Exploring the integer addition and subtraction landscape: Perspectives on integer thinking* (pp. 109-142). Springer. https://doi.org/10.1007/978-3-319-90692-8_5

Sari, P., Hajizah, M. N., & Purwanto, S. (2020). The neutralization on an empty number line model for integer additions and subtractions: Is it helpful?. *Journal on Mathematics Education, 11*(1), 1-16. http://doi.org/10.22342/jme.11.1.9781.1-16

Sari, P. (2023). Developing students' understanding of integer addition and subtraction. *Doctoral Dissertation*. National Institute of Education, Nanyang Technological University.

Schwarz, B. B., Kohn, A. S., & Resnick, L. B. (1994). Positives about negatives: A case study of an intermediate model for signed numbers. *The Journal of the Learning Sciences*, *3*(1), 37-92. https://doi.org/10.1207/s15327809jls0301_2

Shutler, P. M. (2017). A symbolical approach to negative numbers. *The Mathematics Enthusiast, 14*(1), 207-240. https://doi.org/10.54870/1551-3440.1395

Stephan, M., & Akyuz, D. (2012). A proposed instructional theory for integer addition and subtraction. *Journal for Research in Mathematics Education, 43*(4), 428-464. https://doi.org/10.5951/jresematheduc.43.4.0428

Thompson, P. W., & Dreyfus, T. (1988). Integers as transformations. *Journal for Research in Mathematics Education 19*(2), 115-133. https://doi.org/10.5951/jresematheduc.19.2.0115

Ulrich, C. (2012). The addition and subtraction of signed quantities. In L. H. R. Mayes, & M. Mackritis (Ed.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (pp. 127-141). University of Wyoming.

Vig, R., Murray, E., & Star, J. R. (2014). Model breaking points conceptualized. *Educational Psychology Review, 26*, 73–90. https://doi.org/10.1007/s10648-014-9254-6

Vlassis, J. I. (2004). Making sense of the minus sign or becoming flexible in 'negativity'. *Learning and Instruction, 14*(5), 469-484. https://doi.org/10.1016/j.learninstruc.2004.06.012

Vlassis, J. I. (2008). The role of mathematical symbols in the development of number conceptualization: The case of the minus sign. *Philosophical Psychology*, *21*(4), 555-570. https://doi.org/10.1080/09515080802285552

Vosniadou, S., & Verschaffel, L. (2004). Extending the conceptual change approach to mathematics learning and teaching. *Learning and Instruction*, *14*(5), 445-451. https://doi.org/https://doi.org/10.1016/j.learninstruc.2004.06.014

Wessman-Enzinger, N. M., & Mooney, E. S. (2014). Making sense of integers through storytelling. *Mathematics Teaching in the Middle School*, *20*(4), 202-205 https://doi.org/10.5951/mathteacmiddscho.20.4.0202

Wessman-Enzinger, N. M., & Mooney, E. S. (2019). Conceptual models for integer addition and subtraction. *International Journal of Mathematical Education in Science and Technology*, *52*(3), 349–376. https://doi.org/10.1080/0020739X.2019.1685136

Whitacre, I., Bishop, J. P., Lamb, L. L. C., Philipp, R. A., Schappelle, B. P., & Lewis, M. L. (2012). Happy and sad thoughts: An exploration of children's integer reasoning. *Journal of Mathematical Behavior*, *31*(3), 356-365. https://doi.org/10.1016/j.jmathb.2012.03.001