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Advancing future mathematics teachers' geometric thinking through a Van Hiele-based elementary geometry course

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Abstract

Research in mathematics education has increasingly emphasized the importance of developing deep conceptual understanding and higher-order thinking skills in geometry learning. However, traditional approaches to teaching elementary geometry in teacher education programs often remain procedural and insufficiently foster progression through the Van Hiele levels of geometric thinking. Addressing this gap, the present study introduces and examines the method of local axiomatization as a novel instructional approach for preparing future mathematics teachers. The purpose of the study is to identify, characterize, and test practical strategies for teaching an "Elementary Geometry" course through this method, with the goal of facilitating teacher candidates' advancement across the Van Hiele model of geometric thinking. The research highlights effective educational practices, including maintaining student motivation, inquiry-based learning, collaborative interaction, integration of technology, strategic problem-solving, and reflective error analysis. Based on these principles, a university-level course in elementary geometry was designed and implemented as research training for 56 prospective mathematics teachers. Data were collected through the Van Hiele Geometry Test (VHGT), administered before and after the intervention, and through reflective essays written by participants. Statistical analysis using the Pearson criterion demonstrated a significant increase in students' levels of geometric thinking, while qualitative reflections indicated enrichment of geometric knowledge and more independent, yet guided, learning. The findings suggest that the method of local axiomatization, despite implementation challenges, can serve as an effective and innovative pedagogical framework in mathematics teacher education, contributing to the development of both conceptual understanding and reflective practice in geometry learning.

Keywords: Elementary Geometry, Future Mathematics Teachers, Geometric Teacher Training, Geometric Thinking, Local Axiomatization, Teacher Education, Van Hiele Model

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One of the central objectives of contemporary mathematics education is the comprehensive intellectual development of students. Within this framework, the role of the mathematics teacher acquires particular importance, as the effectiveness of cultivating mathematical thinking among school students is largely determined by the teacher's professional and intellectual preparation. Consequently, attention must also be directed toward the development of mathematical thinking in teachers themselves.

A fundamental component of mathematical thinking is geometric thinking, which rests on visualization, spatial reasoning, the manipulation of geometric representations, and the capacity to construct and evaluate logical arguments. Fostering this form of thinking should therefore constitute a key objective in the preparation of prospective mathematics teachers in teacher education programs. When the educational process is effectively organized to promote geometric thinking, it not only deepens students' disciplinary knowledge but also strengthens their pedagogical readiness to teach geometry at the school level. Thus, the formation of mathematical thinking in prospective teachers requires a comprehensive approach, beginning with the deliberate cultivation of such thinking during their own training.

One major factor that impedes the advancement of higher-order thinking in prospective mathematics teachers is the persistence of traditional approaches to teaching geometry. In these approaches, students are presented with pre-structured, fully prepared material, the mastery of which is often reduced to processes of memorization and reproduction. Such instructional practices constrain opportunities for independent analysis, reasoning, and problem solving, thereby limiting the development of both general mathematical thinking and geometric thinking in particular. Considering this challenge, a crucial task in mathematics teacher education is the identification and implementation of instructional strategies and practices that explicitly foster students' capacity for mathematical reasoning and problem solving across university-level courses.

Within the university preparation of prospective mathematics teachers, specialized courses in school mathematics occupy a particularly important place. These courses serve multiple purposes: they address gaps in students' prior knowledge and skills, provide a systematic review of mathematical facts and methods foundational for advanced study, and, crucially, foster the development of mathematical thinking. Despite these intentions, research has consistently shown that many future mathematics teachers exhibit deficiencies in mathematical thinking, and particularly in geometric thinking (Armah et al., 2017; Hourigan & Leavy, 2017; Shkerina et al., 2022).

Although the literature does not offer a single, universally accepted definition of mathematical thinking, scholars generally regard it as a distinctive and integral form of human thought. Mathematical thinking may be understood as a mode of cognition whose thematic content is intrinsically mathematical. For example, Fridman (2005, p. 41) defines it as "*an extremely abstract theoretical thinking, the objects of which are devoid of any substance and can be interpreted arbitrarily, provided that the indicated relations are maintained between them.*" Tall (2019) further identifies abstraction, synthesis, generalization, modeling, problem solving, and proof as central components of mathematical thinking.

In addressing the specific problem of developing geometric thinking, the Van Hiele theory (Van Hiele, 1984) has played a foundational role. Based on its theoretical principles, Dina and Pierre Van Hiele formulated a hierarchical model of geometric reasoning, consisting of five levels that describe the progression of learning. These levels can be briefly characterized as follows:

1. Visual – recognition of figures through holistic visual images, without decomposition into components or relation to other objects.
2. Descriptive/Analytical – ability to associate visual representations with properties of geometric figures.
3. Abstract/Relational – logical organization of shapes and their properties.
4. Deductive – understanding of the axiomatic–deductive method as the basis for constructing and developing geometric theory through formal proof.
5. Mathematical – abstraction beyond specific geometric objects, where geometry is treated as an abstract deductive system.

According to the Van Hieles, a learner's progression through these levels is achieved through the transition from implicit to explicit understanding, with each level characterized by its own mathematical language. This observation underscores the pedagogical importance of teachers recognizing and accounting for such language differences in instruction. Moreover, recent research (Bonyah & Larbi, 2021) highlights that the fifth level of the Van Hiele model often transcends the boundaries of formal instruction and may be practically impossible to assess with precision.

Previous research demonstrates that the Van Hiele model has been employed in diverse contexts within mathematics education. It has served as the basis for constructing tests and assessment instruments (Chen et al., 2019; Usiskin, 1982), diagnosing levels of geometric thinking, including among prospective teachers (Altakhyneh, 2018; Bonyah & Larbi, 2021; Fitriyani et al., 2018; Žilková et al., 2025), designing methods and didactic materials (Santos et al., 2022), and evaluating instructional strategies (Armah & Kissi, 2019; Altakhyneh, 2018; Usman et al., 2020). Many of these studies are highly specialized, either addressing particular mathematical topics (Roldán-Zafra et al., 2022; Sudihartinih & Wahyudin, 2019) or focusing on narrowly defined forms of pedagogical intervention (Al-ebous, 2016; Hassan et al., 2020; Santos et al., 2022). A notable subset of research highlights the role of digital technologies in fostering inquiry-based and research-oriented approaches to geometry education within the framework of the Van Hiele model (Klemer & Rapoport, 2020; Marrades & Gutiérrez, 2025).

Despite these contributions, significant gaps remain. In particular, there is a shortage of studies examining the impact of entire university-level courses on students' progression through the Van Hiele levels of geometric thinking. Likewise, detailed accounts of how geometry courses can be systematically designed to support such progression are scarce. Furthermore, although the method of local axiomatization—introduced in classical works by Freudenthal (1973), Krygowska (1971), and Stoliar (1974)—holds considerable promise as a pedagogical tool for developing geometric thinking, it has received limited attention in both contemporary research and the practice of teacher education. To date, there are virtually no modern studies that investigate its implementation under authentic instructional conditions.

The present study seeks to address these gaps. For the first time, it examines how the method of local axiomatization can be integrated into the teaching of elementary geometry for prospective mathematics teachers, in combination with other effective educational strategies, and how such integration influences the development of their mathematical and geometric thinking. This focus defines the novelty of the research.

Accordingly, the purpose of the study is to identify, characterize, and evaluate practical approaches to teaching the university course Elementary Geometry through the method of local axiomatization, with the aim of supporting prospective teachers' advancement across the Van Hiele levels of geometric reasoning. Specifically, the study addresses the following research questions:

1. What educational practices are most effective for the university preparation of future mathematics teachers?
2. How can the design of an Elementary Geometry course, based on local axiomatization and effective pedagogical practices, foster the development of prospective teachers' mathematical and geometric thinking?
3. What outcomes result from the implementation of such a course in terms of students' progression through the levels of geometric thinking?

METHODS

Research Design

The study employed a quasi-experimental design with pre- and post-testing, implemented within an explanatory sequential mixed-methods framework. At the first stage, quantitative data were collected using the Van Hiele Geometry Test (VHGT) to evaluate changes in participants' levels of geometric thinking. At the second stage, qualitative data were obtained through reflective essays, which served to interpret and enrich the quantitative findings. Integration of the two data strands occurred at the interpretation stage to provide a comprehensive understanding of the impact of the instructional intervention.

The pedagogical practices applied in the study were identified through a reflexive analysis of prior research in mathematics education and mathematics teacher preparation. These practices were validated through several years of implementation in the authors' teaching and further refined through collegial discussions within the university. The design of the Elementary Geometry course was developed directly by the authors.

Participants and Ethical Considerations

The study sample consisted of 56 third-year undergraduate students enrolled in a mathematics teacher education program. All participants were informed about the objectives, procedures, and conditions of the study, and voluntary informed consent was obtained prior to participation. Anonymity and confidentiality were guaranteed: no personal identifiers were collected, and results are reported only in aggregated form. The study was approved by the Institutional Ethics Committee of Kozybayev University and conducted in accordance with international standards of ethical research in education.

Quantitative Instrument: Van Hiele Geometry Test (VHGT)

To assess participants' levels of geometric thinking, the study utilized the Van Hiele Geometry Test (VHGT), originally proposed and validated by Usiskin (1982). This instrument has been widely employed in previous studies on prospective mathematics teachers (e.g., Armah et al., 2017; Halat & Şahin, 2008) and is recognized for its reliability and effectiveness. Consistent with Bonyah and Larbi (2021), the analysis in this study focused on the first four Van Hiele levels, as the fifth level lies beyond the scope of formal instruction.

The test, consisting of 20 multiple-choice items organized into four blocks of five tasks (corresponding to levels 1–4), was translated and adapted for students studying in Kazakh and Russian. Adaptation followed a multistage process: (a) translation and back-translation by two independent bilingual experts to ensure linguistic accuracy, (b) validation by subject-matter experts to ensure cultural and contextual appropriateness, and (c) reliability testing. Cronbach's α was 0.812 for the pre-test and 0.802 for the post-test, indicating high internal consistency.

Scoring followed Usiskin's (1982) "3 out of 5" criterion, whereby a level was considered achieved if at least three tasks in the corresponding block were answered correctly. To determine overall progression, weighted scores were assigned: 0 points if no level was achieved; 1, 2, 4, and 8 points for mastering levels 1 through 4, respectively. This cumulative scoring procedure required that a student meet the criteria for all preceding levels in order to be credited with a higher level of geometric thinking.

Qualitative Instrument: Reflective Essays

At the end of the course, participants were asked to write short reflective essays on their experiences. Guiding prompts included:

1. Did the course influence your mathematical understanding, problem-solving ability, or thinking skills?
2. Did the course improve your geometric thinking? If so, in what ways?
3. Which instructional methods or tools from the course would you adopt in your own teaching to enhance students' geometric thinking?

Responses were voluntary and could be expressed freely without strict adherence to the prompts.

Qualitative Data Analysis

Four experts independently analyzed the essays using NVivo software to ensure coding reliability. The unit of analysis was defined as a sentence or group of sentences expressing a complete thought. Statements were categorized into semantic groups and coded as positive ("+"), negative ("–"), or mixed ("–, +" or "+, –") depending on their evaluative orientation. Representative quotations were retained to illustrate each category. To avoid misinterpretation, all essays were reviewed by each of the authors during the final stage of analysis.

RESULTS AND DISCUSSION

Effective Educational Practices for Training Future Mathematics Teachers

A significant barrier to the development of higher-order thinking in prospective mathematics teachers is the traditional teaching of geometry, which often relies on pre-prepared content requiring only memorization. In selecting effective instructional practices for the Elementary Geometry course, we were guided by the premise that the success of teacher training depends on fostering deep subject-pedagogical knowledge and providing opportunities for active acquisition of both disciplinary knowledge and professional teaching skills (Shmigirilova et al., 2019; Thurlings & den Brok, 2018). The following key aspects were prioritized in the course design to achieve these objectives.

1. Motivation to Learn
   Motivation is widely recognized as a crucial factor influencing learning outcomes (Chen, 2015; Lynch & Trujillo, 2011; Matos et al., 2017). Research indicates a positive correlation between students' motivation, self-efficacy, and performance, particularly when learners perceive tasks as meaningful and valuable (Code et al., 2016; Vishnumolakala et al., 2017). In the context of teacher education, task design that emphasizes relevance and intellectual challenge enhances motivation and engagement.
2. Research-Based and Problem-Based Learning (PBL)
   Student-centered approaches such as research-based and problem-based learning are highly effective in higher education, particularly in teacher preparation programs. These methods encourage active exploration of content and the development of problem-solving skills, both central to mathematical thinking (Hemker et al., 2017; Milner & Scholkmann, 2023; Moreno et al., 2024; Ni et al., 2018). PBL provides a structured yet flexible environment in which prospective teachers engage with authentic mathematical problems, fostering analytical reasoning and reflective practice.

3. Collaboration and Interaction
   Collaborative learning promotes the creation of rich and dynamic educational environments (Taggart & Wheeler, 2023). It supports social interaction, diverse perspectives, and critical thinking, all of which contribute to conscious knowledge construction. Dialogue-based interactions, in which students present, justify, and critique solutions, are particularly effective in developing mathematical thinking (Howe et al., 2019). Additionally, collaboration in designing instructional materials with university lecturers strengthens professional competence and produces high-quality educational resources (Meilinda et al., 2024). These collaborative practices simultaneously cultivate mathematical knowledge and pedagogical awareness, preparing future teachers to implement dialogic learning in their own classrooms.
4. Problem Solving in Mathematics
   Problem solving remains a central component of mathematics and geometry education (Fried, 2014; Santos-Trigo, 2024). Problems with multiple solutions or real-world contexts encourage critical thinking and active engagement. Joint exploration of solution strategies, discussion of alternatives, and selection of optimal approaches stimulate cognitive development and reinforce understanding of mathematical principles (Guberman & Leikin, 2013; Ramlan & Hali, 2018; Smith & Mancy, 2018).
5. Analysis of Decisions and Errors
   Reflection on mistakes is an essential pedagogical tool for developing both professional and mathematical competence. Systematic discussion of errors made during problem solving fosters deeper understanding and promotes meta-cognitive awareness (Dalinger, 2015; Mallue, 2018; Shaughnessy et al., 2021). Properly structured error analysis transforms mistakes into opportunities for learning rather than mere failure.
6. Use of Technology
   The integration of information and communication technologies in mathematics education enhances students' skills, knowledge, and thinking (Clark-Wilson et al., 2020; Hillmayr et al., 2020; Thurm & Barzel, 2020; Koyunlu & Dokme, 2020). When applied thoughtfully, technology creates dynamic and interactive learning experiences that support visualization, exploration, and modeling. Curriculum design should balance opportunities for repeated practice with the development of teachers' digital competencies within pedagogical contexts (Fathurrohman et al., 2017; García-Lázaro & Martín-Nieto, 2023).

Dynamic geometry software, such as GeoGebra, plays a particularly significant role in geometry instruction (Bernard & Setiawan, 2020; Küçük & Gün, 2023; Niroj et al., 2022). Such tools enable dynamic linking of multiple representations and strengthen the figurative component of mathematical thinking. Empirical evidence indicates that dynamic geometry environments can facilitate progression through the Van Hiele levels of geometric reasoning (Karakuş & Peker, 2015; Rvanova, 2018). Importantly, the effectiveness of technology depends on its didactic implementation: tools should be used purposefully to accurately and clearly represent geometric situations, thereby enhancing comprehension and reasoning.

Course Design and Implementation

The Elementary Geometry course was structured according to a multi-stage framework that integrated effective educational practices with the principles of local axiomatization (Freudenthal, 1973;

Krygowska, 1971; Stoliar, 1974). The authors, who were also instructors at the university during the study, designed and directly implemented the course.

1. Stage 1: Identification and Description of Properties

   The first stage involved the independent identification and description of the properties of the geometric situation under study, whether a single figure or a set of figures. Students revisited and consolidated relevant content from school textbooks, guided by preparatory instructional materials containing prompts and questions to facilitate independent work. Examples of guiding questions included: "*Using folding and superimposition, compare the lengths of the sides (or angles) of quadrilateral ABCD*" and "*What properties do the diagonals of quadrilateral ABCD possess?*" Supplemental materials could provide verbal descriptions or diagrams for properties not covered in standard curricula. The outcome of this stage was the construction of a comprehensive set of statements representing the properties of the geometric situation:

   $P = \{p_1, p_2, \dots, p_n\}$.

   For example, for the parallelogram ABCD, where O is the point of intersection of the diagonals, the following list of properties can be specified: p_1: AD||BC; p_2: AB||DC; p_3: AD = BC; p_4: AB = DC; p_5: $\angle$BCD =$\angle$BAD; p_6: $\angle$ABC =$\angle$ADC; p_7: AO = OC; p_8: BO = OD; p_9: O is the center of symmetry of the quadrangle ABCD; p_{10}: OA·OB = OC·OD; p_{11}: OA·OD = OB·OC; p_{12}: $S_{ABD} = S_{ACD}$; p_{13}: $S_{ABD} = S_{ABC}$; p_{14}: $S_{ABO} = S_{CDO}$; p_{15}: $S_{ADO} = S_{BCO}$; p_{16}: $S_{ABO} = S_{CBO}$; p_{17}: $S_{ABO} = S_{ADO}$; p_{18}: $(S_{ABO})^2 = S_{BCO} \cdot S_{ADO}$; p_{19}: $(S_{ADO})^2 = S_{ABO} \cdot S_{CDO}$. The list of properties can be expanded. In addition, properties can be grouped by thematic affiliation. For example, properties related to parallelism; properties using area, etc.

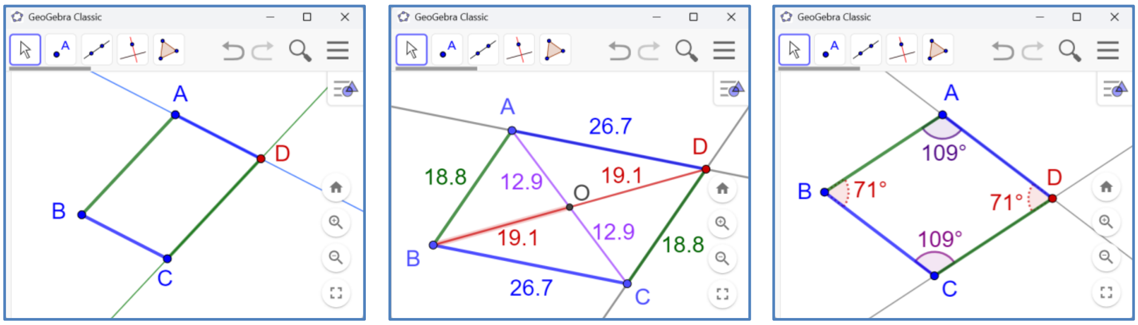
2. Stage 2: Logical Experiment and Construction of Local Axiomatic Systems

   In the course of a logical experiment, the properties of the set P = $\{p_1, p_2, \dots, p_n\}$ are investigated to determine the possibilities of its logical organization. The task is to select from a given set the minimum sets of sentences (systems of local axioms), from which all the sentences of the set P follow. A logical experiment on the set P begins with the fact that the statement p_1 is taken as true, and it is checked whether all the others follow from it. To do this, the statement $p_1 \rightarrow p_i$ is formulated, then this statement is proved, or a counterexample is given, and accordingly a conclusion is made about the truth or falsity of the statement. The statement p_1 is not sufficient, for example, for the statements p_2, p_3, etc. Therefore, we add another statement to it so that the resulting set satisfies the requirements: 1) no other sentence of this set follows from any sentence of the set; 2) all other sentences of the theory follow from the set of sentences. For example, all statements $(p_1, p_2) \rightarrow p_i$ are true, so (p_1, p_2) is a local system of axioms on the set P. In addition, on the set of properties of a parallelogram: P = $\{p_1, p_2, \dots, p_{19}\}$ as the initial ones, we can take the sets of sentences (p_1, p_3), (p_1, p_5), (p_1, p_6), (p_1, p_7), (p_1, p_8), (p_2, p_4), (p_2, p_5), (p_2, p_6), (p_2, p_7), (p_2, p_8), (p_3, p_4), (p_5, p_6), (p_7, p_8), (p_9) , (p_7, p_{17}), (p_{10}, p_{11}), (p_{12}, p_{13}), (p_{14}, p_{15}), (p_{16}, p_{17}), (p_{18}, p_{19}) etc. Each system of local axioms determines the version of the local theory of the parallelogram and the corresponding definition or characteristic of the parallelogram.

   This stage facilitated the updating and deepening of geometric knowledge, as well as the formation of intra-mathematical connections. Mental operations undertaken during logical experimentation corresponded to the third and fourth levels of the Van Hiele model. Pair or group work was encouraged to support reasoning and collaboration.

3. Stage 3: Working with Dynamic Models

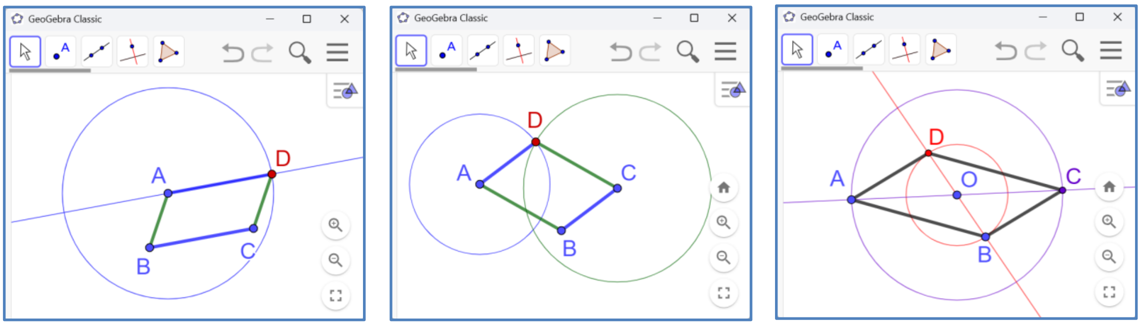
   At this stage, students work in pairs or small groups. For example, when considering parallelograms, students in the dynamic environment of GeoGebra build a dynamic model of a geometric situation by definition using the "Parallel Line" tool (Figure 1 A).



A) B) C)

Figure 1. Dynamic drawings of a parallelogram in GeoGebra

After performing some measurements (Figure 1 B, C), they determine the properties of the parallelogram. Since the model is dynamic, it is useful to allow students to observe changes as it develops. It is also important to discuss with students that they are conducting inductive research, working with a dynamic model, the conclusions of which are plausible, but unreliable and require proof. Then, future teachers are instructed to develop a scenario of how such models can be used to demonstrate to students a set of geometric properties of a situation, which together determine the feature of a given configuration. For example, a parallelogram can be constructed on the basis of the feature of parallelism and equality of two sides (Figure 2 A), on the basis of the feature of equality of pairs of opposite sides (Figure 2 B), on the basis of the feature using diagonals (Figure 2 C), etc.



A) B) C)

Figure 2. Examples of dynamic parallelogram drawings made by students in GeoGebra

Further study of the model involves finding answers to the questions: "*Whether the model always exists in dynamics?*", "*Under what conditions it degenerates and why?*", "*Which model is most appropriate to use depending on the conditions of the problem?*". The stage ends with a collective discussion of these results of local axiomatization.

Thus, this stage solves several didactic problems. Independent work of students in the GeoGebra environment and the inclusion of this technology in contextual learning contributes to their pedagogical understanding of both mathematical and technological content. By interacting with each other when working in pairs or groups, as well as presenting the results of their work, students learn to reason, formulate hypotheses and draw conclusions, which relates to the activity of geometric thinking.

4. Stage 4: Construction and Solution of Problems
   Students were first asked to create tasks independently as part of extracurricular work, utilizing the properties and characteristics of the geometric situation they had previously studied. These tasks included both evidentiary problems derived from the logical experiment stage and novel problems constructed based on the outcomes of that experiment.
   During classroom activities, students worked in pairs, exchanging the tasks they had created. Each student solved the problems provided by their partner, followed by a discussion of the solutions. Errors were analyzed to identify their causes, and a process of mutual evaluation was implemented to foster critical reflection. At the conclusion of the lesson, students received additional sets of tasks from the instructor for independent completion, which could include problems from school textbooks or practical applications. Solutions were subsequently reviewed collectively, with emphasis on analyzing mistakes and highlighting the most effective or innovative solution strategies.
5. Stage 5: Generalization and Reflection
   In this stage, prospective teachers worked collaboratively in groups to create mental maps, employing cognitive operations such as comparison and generalization. These maps captured the content of the studied module and illustrated the relationships between individual concepts and facts. The lesson concluded with a collective reflection, during which students articulated the key insights they had gained and the skills they had developed throughout the module.

Overall, the course emphasized independent work by prospective teachers, supported by specially developed instructional materials, guidance from the instructor to address learning difficulties, and opportunities for collective discussion of various content aspects. Importantly, the results of the Van Hiele pre-test were used to inform the design of these materials, the formation of collaborative learning groups, and the selection of tasks for individual practice. As the course progressed, scaffolding was gradually reduced: instructional materials provided fewer hints, and the complexity of tasks was systematically increased to promote autonomous reasoning and higher-level geometric thinking.

Results of the Empirical Stage

The analysis of the results presented in Table 1 indicates a general trend toward an increased level of geometric thinking following completion of the course. The category “level not defined” included pre-service teachers who demonstrated competencies indicative of a higher level of geometric thinking but whose responses to tasks at the preceding level did not satisfy the scoring criteria. Upon further examination, this discrepancy appears to be primarily attributable to students’ tendency to respond hastily without fully considering all provided options.

Statistical analysis of pre- and post-course test results was conducted using the Pearson chi-square test. Comparison of the calculated values indicates a statistically significant difference in geometric thinking levels among prospective teachers before and after the "Elementary Geometry" course. The empirical chi-square value was 13.76, exceeding the critical value of 11.345 at a significance level of 0.01. The effect size, calculated using Cramér's V, was 0.496, indicating a practically meaningful effect approaching the threshold for strong significance.

Table 1. Levels of geometric thinking achieved by students before and after the course based on VHGT

Level	*Before the Course*	*After the Course*
Level is uncertain	10	6
Level 1-2	16	6
Level 3	18	14
Level 4	12	30
Total	56	56

To validate the educational relevance of the test, follow-up observations of students engaged in various cognitive activities confirmed the test results. Students' reasoning patterns revealed differences in geometric thinking levels consistent with the results, aligning with observations in the literature (e.g., Kuzniak, 2008). Notably, the thinking levels of some students remained uncertain.

Analysis of reflective essays yielded 326 statements suitable for coding. These were subsequently categorized into 34 thematic groups. Table 2 presents a representative fragment of this summary, including example statements from students' essays.

Table 2. Summary analysis of statements from reflective essays of future teachers (fragment)

Group of Statements	*Number of Essays Containing a Statement*	*Representative Statement*	*Code*	*Examples of Statements in the Essay*
▪ Awareness of the importance of further development of geometric thinking for future professional activities	40	I cannot be completely sure that I have a high level of geometric thinking, but I realized that it is important for a teacher to have a high level of such thinking.	–, +	▪ My geometric thinking is not yet high, but this is very important for the work of a teacher. ▪ A math teacher needs to have a high level of geometric thinking, but I doubt that I already have such a level. ▪ I realized that I need to continue to develop my thinking, because it is so important for a math teacher.
▪ Development of understanding of the axiomatic method of constructing a mathematical theory	35	Understood the axiomatic method of constructing a mathematical theory / Formed (changed / corrected the erroneous idea) about the axiomatic method of constructing	+	▪ Studying the course, I realized that the axiomatic method is not only a system of axioms from a textbook, but also a way of constructing a theory. ▪ I finally understood what the axiomatic method in geometry is. ▪ Mastered the axiomatic method and began to better understand how

		a mathematical theory.		geometry works.
▪ Rethinking the axiomatic method as a teaching method	20	He saw that mathematical methods (in particular, axiomatization) can be used as teaching methods.	+	▪ I would be glad if I was taught geometry through local axiomatization at school - it is interesting. ▪ It turns out that the axiomatic method allows you to better understand geometry.
▪ Understanding the need for correct visualization of geometric shapes using technology	41	I am convinced that the demonstration of forms using technology should be carried out correctly from the point of view of geometric concepts.	+	▪ I realized how important it is to build shapes correctly in GeoGebra. ▪ I have seen from my own experience that an incorrectly built GeoGebra can prevent you from seeing the solution to a problem.
▪ Development of logical analysis skills, the ability to detect errors in statements	16	Learned to detect errors in statements and use counterexamples to prove their falsity.	+	▪ By the end of the course, I learned to see mistakes both in solving the problem and in reasoning. ▪ He realized that counterexamples are an important tool for proving the falsity of statements and learned to come up with them.
▪ Development of skills to correlate a geometric image with the properties of a figure	11	I realized that often a geometric problem can be solved if, instead of the term denoting a figure, we refer to its definition or properties.	+	▪ Now, when I hear the name of a figure, I imagine not only its image, but also the properties it possesses. ▪ I realized that if you find it difficult to solve a problem, you need not just look at the drawing but remember the properties of the figures mentioned in the problem.
▪ Focus on the use of the method of local axiomatization in future professional activities	10	I would like to use the technique of local axiomatization when working with schoolchildren, but I am not yet sure that I can.	+, –	▪ I think that the method of local axiomatization can be used in school, but I would like to master it better. ▪ I need to better understand the method of local axiomatization, if I must work with gifted students, then I will use it.

The average Cohen's Kappa coefficient for the four experts analyzing the essays is presented in Table 3.

Table 3. Average Pairwise Cohen's Kappa

Average pairwise CK	*Pairwise CK cols 1 & 4*	*Pairwise CK cols 1 & 3*	*Pairwise CK cols 1 & 2*	*Pairwise CK cols 2 & 4*	*Pairwise CK cols 2 & 3*	*Pairwise CK cols 3 & 4*
0.845	0.826	0.845	0.867	0.848	0.86	0.822

Table 4 presents some semantic expressions of statements obtained during the analysis of the essay, which were not always the most common, but to a greater extent related to the problem under study. In addition, the reflective essays contained statements pertaining to the self-perception of pre-service teachers during the process of studying the course. For instance, several participants emphasized the development of confidence and autonomy in their mathematical activity: "*I felt that I was not afraid to express my ideas*," "*I was happy when I was able to independently solve a problem that at first seemed difficult*," "*I was proud when I was able to formulate the definition of a figure myself when constructing a local theory*," and "*I began to feel more confident in my knowledge of school geometry*."

Table 4. Analysis of the results of the reflective essay

Coding Sign	Meaning of the Statement	Percentage of Essays Containing a Statement (Rounded to Hundredths)
–, +	▪ I cannot be completely sure that I have a high level of geometric thinking, but I realized that it is important for a teacher to have a high level of such thinking.	71.43
–, +	▪ The stage of the logical experiment seemed to me at first very difficult, I was not ready to cope with it, it turned out when we thought together and discussed in the group.	37.50
+	▪ Understood the axiomatic method of constructing a mathematical theory / Formed (changed / corrected the erroneous idea) about the axiomatic method of constructing a mathematical theory.	62.50
+	▪ He saw that mathematical methods (in particular, axiomatization) can be used as teaching methods.	35.71
+	▪ Learned to detect errors in statements and use counterexamples to prove their falsity.	28.57
+	▪ I began to find a strategy for a faster solution to the problem.	60.71
–, +	▪ It was not always possible to solve the problem right away, but I learned to continue looking for a solution, even if it was difficult.	23.21
+	▪ I am convinced that the demonstration of forms using technology should be carried out correctly from the point of view of geometric concepts.	73.21
+	▪ I realized that often a geometric problem can be solved if, instead of the term denoting a figure, we refer to its definition or properties.	19.64
+	▪ I began to better understand how a geometric problem works.	21.43
–,+	▪ It was not possible to prove and correctly formulate the conclusions, this happened by the end of the course.	41.07
+, –	▪ I would like to use the technique of local axiomatization when working with schoolchildren, but I am not yet sure that I can.	17.86
+	▪ I will definitely use GeoGebra in teaching schoolchildren.	89.29

Furthermore, the essays included reflections related to particular learning activities and personal experiences encountered in the course. Representative statements include: "*Before this course, I did not think that the types of quadrilaterals were related to their properties*," "*The materials and instructions provided by the instructor were very useful*," "*To better understand the problems and facilitate their solution, I decided to construct shapes in GeoGebra*," and "*I got a special notebook to record ideas and materials that I plan to use in my future teaching*."

All participants, in one form or another, reported that completing the course had a positive impact on their geometric thinking. Notably, the reflective essays contained virtually no explicit negative evaluations of the course. When critical remarks were expressed, they were typically balanced with positive reflections and therefore were coded as "–, +" or "+, –." Overall, the essay analysis confirms that the course fostered both the enrichment of geometric knowledge and the development of geometric thinking, particularly in the ability to solve geometric problems. These conclusions were corroborated by classroom observations. Importantly, the essays also provided teacher candidates with opportunities to reflect on their own cognitive processes, on the role of geometric thinking in fostering student success, and on pedagogical methods and technologies they may apply in their future professional practice. Of particular significance is the recurring reference to the method of local axiomatization across diverse contexts, underscoring its central role in shaping students' perceptions of the course.

A major outcome of the study is the restructured design and organization of the elementary geometry course. Persistent obstacles to the development of geometric thinking include the predominance of reproductive teaching practices and the uniform application of instructional methods regardless of students' cognitive levels. Such practices hinder the advancement of students with higher levels of thinking while simultaneously creating substantial difficulties for those at lower levels, often resulting in diminished interest in the subject (Armah et al., 2017; Shaughnessy et al., 2021). The revised course sought to address these challenges through the integration of effective educational strategies.

The course design incorporated a clear sequence of stages for each module: beginning with the revision and systematization of school geometry, progressing through exploratory activities, engaging with a dynamic geometry environment, solving problems, and culminating in generalizing tasks. This sequence aimed to support the systematic development of students' geometric knowledge and understanding. Furthermore, results of the Van Hiele test administered prior to the course were used to tailor instructional materials, facilitate group formation for collaborative learning, and select tasks appropriate for individual work. As the course advanced, scaffolding was gradually reduced and task complexity increased. This approach aligns with findings from prior research (Alex & Mammen, 2016; Armah et al., 2017), which emphasize that geometry courses designed in accordance with levels of geometric thinking provide more effective support for learners' progression. In this regard, the present study contributes to ongoing efforts to improve pre-service teacher education (Hemker et al., 2017; Milner & Scholkmann, 2023; Shkerina et al., 2022) and extends current knowledge of instructional interventions that foster geometric thinking (Altakhyneh, 2018; Hassan et al., 2020; Usman et al., 2020).

A distinctive feature of this study is the integration of the local axiomatization method, which differentiates the course from existing approaches and highlights its impact on the development of pre-service teachers' thinking. Because the implementation of local axiomatization is consistent with principles of research-based learning, this work expands the methodological framework of geometry education by complementing established approaches (Koyunlu & Dokme, 2020; Moreno et al., 2024; Ni et al., 2018).

Nevertheless, the implementation of the course design presented challenges for teacher candidates, particularly during the second stage, which emphasized logical experimentation and the construction of a local axiomatic theory. This phase, structured as a research-based learning experience, required carefully prepared instructional materials and ongoing teacher support in the classroom. Despite extending beyond the initially planned timeframe in the first module, some students were able to complete this stage independently by the end of the course. These findings suggest that the local axiomatization method should be combined with scaffolding strategies (Awi et al., 2024), supported by digital technologies (Clark-Wilson et al., 2020; Sunzuma, 2023). For effective instruction, it is recommended that such courses begin with an assessment of students' thinking levels, enabling instructional design to align with their capabilities. However, it is equally important to recognize that students' cognitive development is dynamic: instructional guidance should be gradually reduced and task complexity progressively increased.

The findings also reinforce conclusions from previous research (Bernard & Setiawan, 2020; Kutluca, 2013; Marrades & Gutiérrez, 2025) regarding the critical role of dynamic geometry environments in supporting geometric thinking. Many students explicitly noted in their essays their intention to integrate such technologies into their future teaching practice. Finally, the study highlights the value of collaborative learning for consolidating conceptual understanding of geometric content. Peer interaction facilitated the exchange and justification of ideas, critique of misconceptions, construction of counterexamples, and evaluation of problem solutions, thereby deepening conceptual grasp (Dalinger, 2015; Howe et al., 2019). Collaboration also proved particularly effective when applying the method of local axiomatization, further confirming its pedagogical potential in teacher education.

CONCLUSION

This study contributes to the growing body of knowledge on the components of teacher education that foster the development of geometric thinking in pre-service mathematics teachers, as conceptualized within the Van Hiele model. The findings highlight the effectiveness of research and problem-based learning, the cultivation of motivation, collaborative learning, the use of digital technologies, and systematic engagement with problem-solving strategies and error analysis.

A key outcome of the study is the development and empirical validation of a scenario for the Elementary Geometry course that integrates these practices. Within this design, the method of local axiomatization plays a central role, supporting the structured advancement of students toward higher levels of geometric thinking. Evidence for this progression was provided by both the Van Hiele Geometry Test (VHGT) results and the analysis of reflective essays.

The theoretical framework of the study, together with the tested course scenario, may serve as a foundation for designing instructional procedures aimed at strengthening the geometric knowledge and thinking of future mathematics teachers. Moreover, the ideas embedded in the course design can be adapted and scaled across cultural and institutional contexts, with potential applicability not only in pre-service teacher education but also in the teaching of school students, particularly those with strong mathematical potential.

The study underscores the pedagogical potential of local axiomatization as both a method for teaching geometry and a means for cultivating geometric thinking. Further research is therefore required to explore methodological aspects of implementing local axiomatization in pre-service teacher education. Promising directions include cross-cultural investigations of the method's applicability within diverse educational systems and examinations of its impact on specific cognitive skills. Additionally, exploring the integration of local axiomatization with scaffolding strategies appears to be a productive avenue for supporting students in constructing local theories.

Despite its contributions, the study has several limitations. First, the relatively small cohort of participants may restrict the generalizability of the findings. Second, contextual factors such as teaching practices, individual characteristics of teacher candidates, and the instructional approaches of course instructors could have influenced the outcomes. Finally, situational aspects inherent to the study design—such as the novelty of the educational setting and the heightened enthusiasm of the instructor—may have had a minor but noteworthy effect on the results.

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