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DEDUCTIVE OR INDUCTIVE? PROSPECTIVE TEACHERS' PREFERENCE OF PROOF METHOD ON AN INTERMEDIATE PROOF TASK

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Abstract

The emerging of formal mathematical proof is an essential component in advanced undergraduate mathematics courses. Several colleges have transformed mathematics courses by facilitating undergraduate students to understand formal mathematical language and axiomatic structure. Nevertheless, college students face difficulties when they transition to proof construction in mathematics courses. Therefore, this descriptive-explorative study explores prospective teachers' mathematical proof in the second semester of their studies. There were 240 pre-service mathematics teachers at a state university in Surabaya, Indonesia, determined using the conventional method. Their responses were analyzed using a combination of Miyazaki and Moore methods. This method classified reasoning types (i.e., deductive and inductive) and types of difficulties experienced during the proving. The results conveyed that 62.5% of prospective teachers tended to prefer deductive reasoning, while the rest used inductive reasoning. Only 15.83% of the responses were identified as correct answers, while the other answers included errors on a proof construction. Another result portrayed that most prospective teachers (27.5%) experienced difficulties in using definitions for constructing proofs. This study suggested that the analytical framework of the Miyazaki-Moore method can be employed as a tool to help teachers identify students' proof reasoning types and difficulties in constructing the mathematical proof.

Keywords: deductive-inductive reasoning, proving difficulties, mathematical proof, prospective teachers

Abstrak

Memunculnya bukti matematika formal merupakan komponen penting dalam mata kuliah matematika tingkat lanjut. Beberapa perguruan tinggi telah mengubah mata kuliah matematika dengan memfasilitasi mahasiswa untuk memahami bahasa matematika formal dan struktur aksiomatik. Namun demikian, mahasiswa menghadapi kesulitan ketika mereka beralih ke konstruksi pembuktian dalam mata kuliah matematika. Oleh karena itu, penelitian ini bertujuan untuk mengeksplorasi bukti matematika calon guru di perkuliahan Semester 2. Metode yang digunakan dalam penelitian ini adalah penelitian deskriptif-eksploratif. Partisipan dalam penelitian ini adalah 240 calon guru matematika di sebuah universitas negeri di kota Surabaya, Indonesia. Respons mereka dianalisis menggunakan kombinasi metode Miyazaki-Moore. Metode ini mengklasifikasikan jenis penalaran yang dilakukan, yaitu deduktif dan induktif dan jenis kesulitan yang dialami selama proses pembuktian. Beberapa hasil menunjukkan bahwa 62,5% dari calon guru menggambarkan penalaran deduktif sedangkan sisanya menerapkan penalaran induktif. Selain itu, hanya ada 15,83% dari jawaban yang diidentifikasi sebagai jawaban yang benar, sedangkan jawaban yang lain menunjukkan kesalahan terkait konstruksi bukti. Kami menemukan bahwa sebagian besar respons calon guru (27,5%) mengalami kesulitan dalam menggunakan definisi dalam membangun bukti. Untuk saran, kerangka kerja analitik metode Miyazaki-Moore dapat digunakan sebagai alat yang bermanfaat bagi guru untuk mengidentifikasi tipe-tipe tipe bukti penalaran siswa dan kesulitan dalam membangun bukti matematika.

Kata kunci: penalaran deduktif-induktif, kesulitan dalam pembuktian, bukti matematis, calon guru

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The construction of formal mathematical proof is an important component of advanced mathematics courses for undergraduate degree (Shaker & Berger, 2016). In recent years, some universities have transformed mathematics courses by introducing the transition of proof or introduction to mathematical reasoning courses

(Selden & Selden, 2007; Smith, 2006), which facilitates college students to understand formal mathematical language and axiomatic structure. However, Clark and Lovric (2008) explored challenges faced by college students as they make the transition to proof construction in mathematics courses. This transition requires college students to change their types of reasoning, for instance, shifting the informal language to formal one, reasoning from mathematical definition, understanding and applying the theorem, and making connections between mathematical objects (Clark & Lovric, 2008).

In addition, college students, including prospective teachers, are also demanded to conceive several skills: a) recognizing reasoning and proof as fundamental aspects of mathematics, b) making and investigating allegations of mathematical conjectures, c) developing and evaluating mathematical arguments and proofs, and d) selecting and using different types of reasoning and methods of proof (National Council of Teachers of Mathematics [NCTM], 2000). Blanton, Stylianou, and David, (2003) agreed that college students need to develop required proving skills to construct a proof. In this case, teachers' knowledge about proof must be given to students because that can help the students strengthen the concept and skill of proof (Carrillo, et al, 2018; Stylianides, 2007). Such skills, more particularly, are also necessary for prospective teachers due to the teacher's need for perceiving a deep understanding of nature and the role of proof for conducting instructional practices (Jones, 1997). Moreover, the math teachers' rationales beyond teaching proof and proving in schools are due to the fact that students have experienced similar reasoning to the mathematicians, such as learning a body of mathematical knowledge and gaining insight about why assertions are true. They also can teach students logical thinking, communication, and problem-solving skills in mathematics.

Although proving is an important part of advanced mathematics, many studies indicated that students often have difficulties in constructing a proof (Moore, 1994; Selden, Benkhalti, & Selden, 2014; Selden & Selden, 2007). Epp (2003) reported that a 'poor' mathematical proof process is caused by the lack of proof-writing attempts. In addition, Moore (1994) carried out an observation of some students' transition to college in which most of them stated that they only memorized the proof since they did not understand proof and how to write it. Furthermore, Edwards and Ward (2004) said that the students could not use mathematical definitions or construct the relation between every day and mathematical languages.

In connection with examining student's mathematical proof, Miyazaki and Moore methods might have inspired many researchers for analyzing student's proof with particular objectives. For example, Kögce, Aydin and Yildiz (2010) adopted Miyazaki's (2000) classification of proof to investigate high school students' level of proof based on types of reasoning. Furthermore, Ozdemir and Ovez (2012) looked for the relationship between prospective teachers' perception proof types proposed by Almeida (2000) and their proving processes related to the experienced types of difficulties (Moore, 1994). In relation with students' common error and misconceptions in mathematical proving associated with the use of Moore's error category of proof, Stavrou (2014) found that the students did not necessarily understand the content of relevant definitions or how to apply them in writing proofs. Another study found that students got difficulties in creating definitions that conformed their concept images or accepted definitions of basic concepts (Dickerson & Pitman, 2016). While those studies concern on a single objective on how students' proof is assessed, the present study highlights that individuals' performance regarding mathematical proof can be explained from at least two aspects namely types of reasoning they involve (Miyazaki's method) and types of difficulties they experience during the proving process (Moore's method). Hence, the researchers argue that obtaining data on both aspects simultaneously leads a broader and more likely fruitful knowledge on how individuals, prospective teachers rather, deal with mathematical proof. Therefore, the present study aims to explore the prospective teachers' proof regarding their types of reasoning and difficulties in a mathematical proof.

Proof is an important aspect in mathematics because it is the main component in understanding mathematics (Kögce et al., 2010) and mathematical thinking (Hanna et al., 2009). Consequently, learning mathematics by mastering mathematical proof along with how to construct it becomes a strategic view (Balacheff, 2010). De Villiers (1990) and Knuth (2002a) stated that the role of mathematical proof is to verify the correctness of a result or truth of a statement, to communicate mathematical knowledge, and to apply an axiomatic system. Its purpose helps investigate the trueness or falseness of an argument regardless the cases and conditions (Baki, 2008) and shows the relevance of the justifications (Lee, 2002).

There are two universally recognized proving methods namely deduction and induction (Kögce et al., 2010; Miyazaki, 2000). Deduction method of proof involves several methods encompassing direct proof, proof by contraposition, and proof by contradiction (Baki, 2008; Moralı et al., 2006). Deduction method in mathematics begins with a general statement or hypothesis and examines the possibilities to reach a specific logical conclusion (Morris, 2002). Induction method is generally used by 8th grade students or secondary school students because they have already learned to prove numerical or geometrical proposition (Miyazaki, 2000). These two methods are based on the types of reasoning used by someone in carrying out a proving process, of which each respectively refers to deductive reasoning and inductive reasoning. Deductive reasoning is unique because it is a process of deducing conclusions from known information (premise) based on formal logic rules, where the conclusions must come from information provided and do not need to validate them with experiments (Ayalon & Even, 2008). Whereas, Christou and Papageorgiou (2007) conveyed that inductive reasoning is a reasoning process from specific premises or observations to reach a general conclusion or an overall rule. Of those two, deductive reasoning, which is used in a deductive proof, is considered the preferred tool in many mathematical communities to verify mathematical statements and demonstrate universality. Therefore, Ayalon and Even (2008) argued that deductive reasoning is often used as a synonym for mathematical thinking.

Knowledge of mathematical proof is considered as one essential component of subject matters (Shulman, 1986) in which mathematics teachers must acquire. Jones (1997) argued that teachers will have an extremely secured subject knowledge base of mathematical proof if they teach it accurately and confidently.

Most researchers who have examined teachers' knowledge of proof have centered on teachers' acceptance of empirical versus deductive arguments as valid proofs. Knuth (2002b) investigated sixteen in-service secondary school mathematics teachers' knowledge about what constituted a proof. Martin and Harel (1989) assessed the notions of proof performed by 101 pre-service elementary school teachers by giving their participants some statements accompanied by predetermined arguments and asking them to rate for revealing the validity. Both Knuth (2002b) and Martin and Harel (1989) concluded that most teachers correctly identified a valid argument and also wrongly accepted invalid arguments as proofs. Some pre-service elementary teachers accepted empirical arguments as proofs (Martin & Harel, 1989; Morselli, 2006; Simon & Blume, 1996).

Teacher's knowledge about proof is indeed not limited to understanding how to construct valid proofs. More broadly, it is related to the knowledge of both content and pedagogical aspects of proof. Steele and Rogers (2012) propose the so-called 'Mathematical knowledge for teaching proof' that can be considered as a meaningful framework to assess teacher's knowledge about proof. Steele and Roger (2012) also mention components of proof knowledge comprising knowledge of defining proof, identifying proofs and non-proofs, creating mathematical proofs, and understanding the roles of proof in mathematics. The first three components give main features on the content knowledge of mathematical proof, while the last component gives attention about the work of teaching proof since it is related to, for instance, checking or confirming students' thinking on the truth of a known idea, unpacking students' thinking and reasoning beyond their decision why a statement is true, confirming students' conjectures, and developing students' new mathematical ideas.

This study focuses on one of the teacher's knowledge, particularly on studying prospective teachers' proof identification and creating mathematical proofs through recognizing them across representations in case of two types of reasoning namely deductive and inductive reasoning and types of difficulties during a proving process.

METHOD

This study used descriptive-explorative research design to explore prospective teachers' types of reasoning and difficulties in carrying out a mathematical proving process. There were 240 prospective teachers who studied at Department of Mathematics of a state university in Surabaya, Indonesia, as the research participants. They were in the first semester so that the present study used convenience sampling method (Miles & Huberman, 1994). The participants consisted of 216 female and 24 male students (aged around 18 years old in average). The data were collected in three years (from 2013 to 2015) by providing initial tests to the participants, as many as 80 prospective teachers per academic year. There were 2 classes per academic year, in which each class consisted of 40 prospective teachers.

Even though Moore's (1994) research used interview to see errors in mathematical proof, the present study tended to examine more on deductive and inductive proof without employing interview like what Miyazaki (2000) did. The data were collected using a simple task of constructing one mathematical proof "*Prove that the sum of two odd numbers is an even number*". Actually, the task type could be more than one, such as *the sum of two even numbers is even, the sum of odd and even numbers is odd*, or *the subtraction variation from two even, odd, or even-odd numbers*. However, the main point of this study was a proof method whether using deductive common symbols or certain numbers that tended to be inductive. This, therefore, only required one sufficient problem determined to represent the use of deductive and inductive reasoning. This simple task consisted of a question that was similar to the task used by Özer and Arõkan (2002) in constructing proof in accordance with the types of reasoning. Instead of only types of reasoning, the present study's analysis also concerned on exploring prospective teachers' proof based on the types of difficulties. Moreover, this task was selected since it was often found in Indonesian secondary school curriculum, which had been frequently learned by prospective teachers who studied early mathematical proof. The process of data collection was carried out during the beginning of number theory and elementary algebra courses in their study. Each prospective teacher had been given 15 minutes to complete the task. Afterwards, each prospective teacher's responses were assessed to investigate the prospective teacher's comprehension of their deductive and inductive knowledge in constructing a proof along with their difficulties in constructing their proofs.

Table 1. Moore's types of difficulties in performing mathematical proof (Moore, 1994)

Type of difficulties	Discussion
D1	Prospective teachers did not know the definitions. That is, they were unable to state the definitions.
D2	Prospective teachers had a little intuitive understanding of the concepts.
D3	Prospective teachers' concept images were inadequate for doing the proofs.
D4	Prospective teachers were unable, or unwilling, to generate and use their own examples.
D5	Prospective teachers did not know how to use definitions to obtain the overall structure of proofs.
D6	Prospective teachers were unable to understand and use mathematical language and notation.
D7	Prospective teachers did not know how to start making a proof.

The data of participants' responses about proof correctness were analyzed by using Miyazaki's (2000) classification for types of reasoning in mathematical proof and Moore's (1994) classification for types of errors in a mathematical proof. Table 1 and Table 2 show the scheme of proof analysis used to analyze prospective teachers' proof.

Table 2. Miyazaki's types of reasoning in performing mathematical proof (Miyazaki, 2000)

Representation	Contents	
	Inductive reasoning	**Deductive reasoning**
Functional language used for demonstration	Proof D	Proof A
Other languages, drawings, and/or manipulated objects	Proof C	Proof B

Proof A was the type of proof when deductive reasoning was involved and a functional language was used in the course of making a proof. Proof was the type of proof where deductive reasoning was involved and other languages, drawings, and movable objects were used in the course of making a proof. Proof C was the type of proof where inductive reasoning was involved and other languages, drawings, and movable objects were used. Proof D was the type of proof where inductive reasoning was involved and a functional language was used.

Table 3. A combined classification of Moore-Miyazaki category of proof

Classification								
Miyazaki ***(Type of reasoning)***		**Moore** ***(Type of difficulties)***						
Type	**Correct Proof**	**Conceptual understanding**					**Mathematical languages and notations**	**Getting started on a proof**
		D1	**D2**	**D3**	**D4**	**D5**	**D6**	**D7**
Proof A	%	AD1	AD2	AD3	AD4	AD5	AD6	AD7
Proof B	%	BD1	BD2	BD3	BD4	BD5	BD6	BD7
Proof C	%	CD1	CD2	CD3	CD4	CD5	CD6	CD7
Proof D	%	DD1	DD2	DD3	DD4	DD5	DD6	DD7

In combining the entries presented in Table 1 and Table 2, the present study applied the coding category presented in Table 3 for conducting an analysis. For example, since Proof A referred to a proof involving deductive reasoning with functional language used for demonstration, and D1 referred to a proof difficulty indicated by prospective teachers' misconception on the definition along with their inability to state the definition, then AD1 referred to a proof performance involving deductive reasoning that indicated difficulties shown by the inability to state definitions. Furthermore, the column correct proof portrays that percentage of the students' correct proofs.

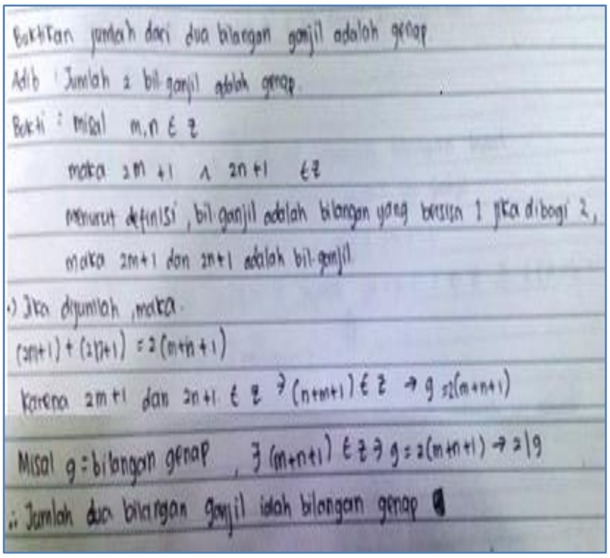
In addition, the coding was carried out to all the prospective teachers' answers. Since there was more than one possibility of coding given to each answer with different categories, the present study selected the most significant feature of the response category that emerged from the answer. Henceforth, each answer only had one code of category. The coding was carried out by the first author and the reliability of the coding was checked through additional coding by an external coder, who was a teacher educator in our university. It was done based on 20 % of 240 prospective teachers' responses in problem proof. 20% of the population chosen randomly became the minimum sample size used in this study that were determined by using Slovin formula with a 10% error margin. In agreement with the multiple coding procedures, this study calculated the inter-rater reliability for each type, which resulted in Cohen's Kappa of 0.69, indicating that the coding was a substantial agreement (Landis & Koch, 1977).

RESULTS AND DISCUSSION

In this section, the data obtained from the participants were analyzed, discussed, and then presented in Table 4. In accordance with Table 4, there were 38 prospective teachers who were correct in mathematical proof, whereas, the others were wrong in mathematical proof. Table 4 also depicts that 61.66% of the prospective teachers performed Proof A in which this proof required deductive reasoning and functional language used to construct proofs. Meanwhile, 0.84% of the prospective teachers conveyed Proof B with deductive reasoning and manipulated objects or using a sentence without functional language in proof. 31.25% of the prospective teachers showed Proof C in which they used inductive reasoning and other languages, images, and manipulated objects to construct proofs. Moreover, 6.25% of the prospective teachers showed Proof D in which they used inductive reasoning and functional language for constructing proofs. Regarding the correctness of the prospective teacher's responses, the present study found that 15.8% of the prospective teachers' responses were correct and 84.2% of the prospective teachers still experienced difficulties in constructing the proof task. Figure 1 to Figure 15 explain the examples of the results of prospective teachers' proof based on Proof A, Proof B, Proof C, and Proof D.

Proof A

The results showed that Proof A was performed by as many as 15.83% of the prospective teachers, meaning that they worked on the proof task correctly according to the deductive reasoning and functional language in constructing proofs. Meanwhile, 45.83% of the prospective teachers had difficulties in proving caused by several things encompassing less understanding of the concept involved (42.08%), lack of knowledge related to mathematical notations (0.42%), and being stuck in starting the proving process (3.33%). In connection with the less understanding of the concept, the prospective teachers' responses consisted of AD1, AD2, AD3, AD4, and AD5 types, while the difficulties to get started on proving included AD7 type, and the lack of knowledge about mathematical notation and logic included AD6 type. Figure 1 shows the correct examples of a prospective teacher's answer to Proof A in constructing the proof.


Buktikan jumlah dari dua bilangan ganjil adalah genap
Adib : Jumlah 2 bil ganjil adalah genap.
Bukti : misal m,n ∈ Z
maka 2m+1 ∧ 2n+1 ∈ Z
menurut definisi, bil ganjil adalah bilangan yang bersisa 1 jika dibagi 2,
maka 2m+1 dan 2n+1 adalah bil ganjil
∴) Jika dijumlah, maka
(2m+1) + (2n+1) = 2(m+n+1)
karena 2m+1 dan 2n+1 ∈ Z ∋ (n+m+1) ∈ Z → g = 2(m+n+1)
Misal g = bilangan genap, ∃ (m+n+1) ∈ Z ∋ g = 2(m+n+1) → 2|g
∴ Jumlah dua bilangan ganjil adalah bilangan genap ■

Translation

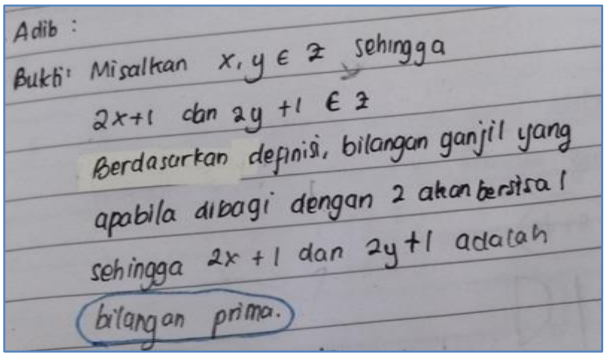
Prove that the sum of two odd numbers is an even number.

It will be proven: the sum of two odd numbers is an even number.

Proof: If m, n ∈ Z then 2m+1 and 2n+1 ∈ Z. According to definition, an odd number is number that has remainder 1 when divided by 2, and then 2m+1 and 2n+1 are odd numbers. If being summed, then (2m+1) + (2n+1) = 2(m+n+1). Because 2m+1 and 2n+1 ∈ Z ∋ (n+m+1) ∈ Z → g = 2(m+n+1). By the definition, an even number is a number that is dividable by 2, then (m+n+1) is an even number.

Figure 1. Example of Proof A

Figure 1 shows that prospective teachers worked on proving with the correct response toward the question given. The correct response in this proof, including Proof A, used deductive reasoning and functional language that could be seen in the student work sample (see Figure 1). The deductive reasoning was indicated by the prospective teacher's idea in firstly letting an even number and an odd number with different symbols, which indicated his understanding of the rigorous symbol that had a significant step for being manipulated in the subsequent proving process. Moreover, it also showed some functional languages precisely, such as the symbols of ∈, Z, ∋ and | that indicated their proficiency in dealing with mathematical symbols.


Adib :
Bukti: Misalkan x, y ∈ Z sehingga
2x+1 dan 2y+1 ∈ Z
Berdasarkan definisi, bilangan ganjil yang
apabila dibagi dengan 2 akan bersisa 1
sehingga 2x+1 dan 2y+1 adalah
bilangan prima.

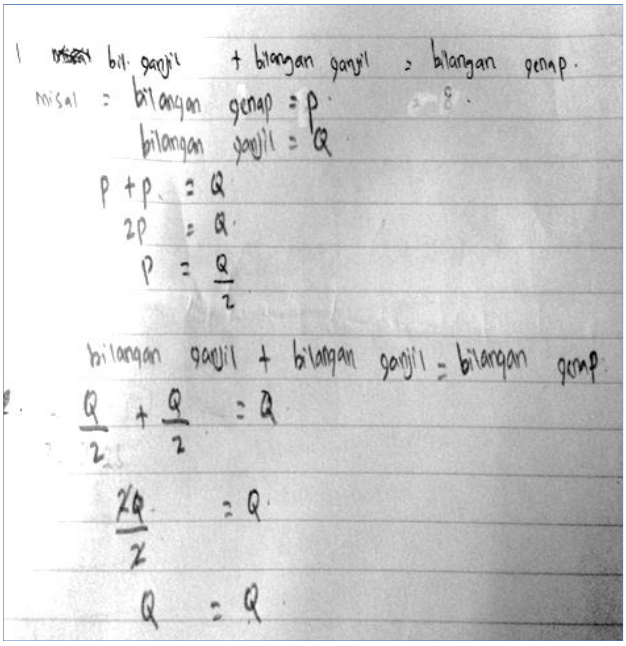
Translation:

It will be proven:

Proof: suppose x,y ∈ Z so that 2x+1 and 2y+1 ∈ Z. By the definition, if an odd number is divided by 2, it will have remainder 1. So, 2x+1 and 2y+1 are prime.

Figure 2. Example of Proof AD1

Figure 2 explains that the prospective teacher's concept in constructing a proof was not well understood, it could be seen that he could not state the definition correctly. It also indicated in the definition of 2x + 1 and 2y + 1 when he wrote as "prime number". Whereas, based on the definition, the form of '2x + 1'and '2y + 1' were an odd number. By definition, an odd number was a number that had a remainder of 1 when divided by 2 so 2x + 1 and 2y + 1 were an odd number. Based on the data, 6.25% of the total prospective teachers using such typical proving process.



Translation:

Odd number + odd number = even number

Suppose: even number = P

odd number = Q

P + P = Q

2P = Q

P $= \frac{Q}{2}$

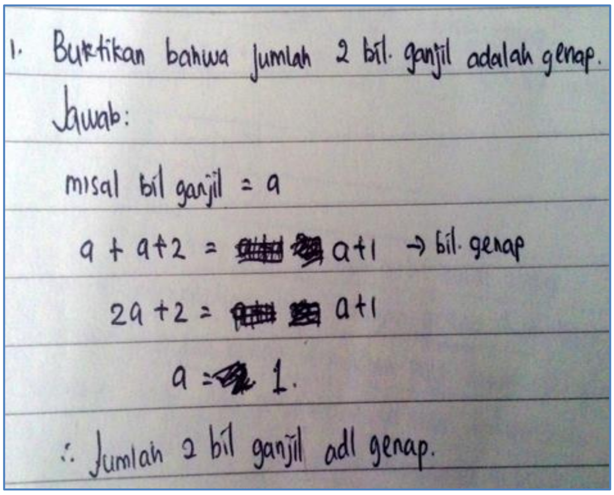
Odd number + odd number = even number

$\frac{Q}{2} + \frac{Q}{2} = Q$

$\frac{2Q}{2} = Q$

Figure 3. Example of Proof AD2

Figure 3 shows that the prospective teacher was not able to construct proofs because of less understanding of the theorem or the concepts involved. The concept of proof that should be proved was used in the proof. The result of $\frac{Q}{2} + \frac{Q}{2} = Q$, which $\frac{Q}{2}$ should have been proved because it was an even number and it was used in the proof. This happened due to prospective teacher's weak intuitive understanding before starting the proving process, so that, she could not solve proof task formally, logically, and relevantly to the definition. Based on the data, 3.75% of the 240 prospective teachers experienced such proving errors.



Translation

Prove that the sum of two odd numbers is an even number.

Answer: suppose odd number = a

a+a+2 = a+1 → an even number

2a+2 = a + 1

a = 1

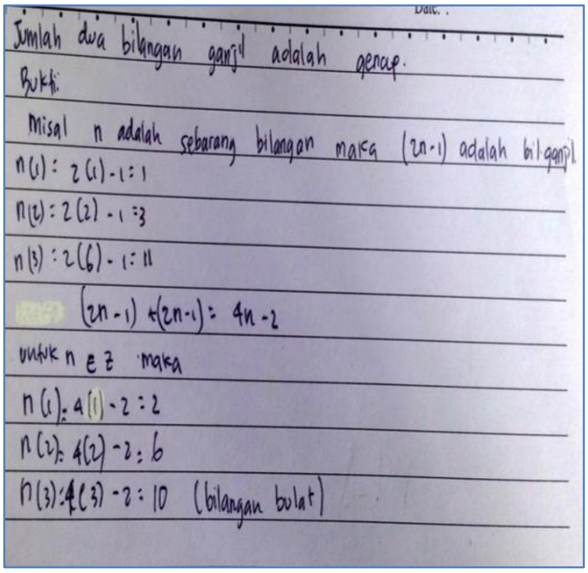
Thus, the sum of two odd numbers is an even number.

Figure 4. Example of Proof AD3

Table 4. The classification of prospective teachers' answers toward the proof task

Classification																	Total	
Miyazaki *(Types of reasoning)*			Moore *(Types of difficulties)*															
Type	Correct Proof		Conceptual understanding										Mathematical language and notation		Getting started on a proof			
			D1		D2		D3		D4		D5		D6		D7			
	n	%	n	%	n	%	n	%	n	%	n	%	n	%	n	%	n	%
Proof A	38	15.8	15	6.25	9	3.75	6	2.5	5	2.08	66	27.5	1	0.42	8	3.33	148	61.66
Proof B	0	0	1	0.42	0	0	0	0	0	0	0	0	1	0.42	0	0	2	0.84
Proof C	0	0	0	0	6	2.5	6	2.5	63	26.25	0	0	0	0	0	0	86	31.25
Proof D	0	0	12	5	0	0	0	0	0	0	3	1.25	0	0	0	0	4	6.25
Number (n)	38	15.8	28	11.6	15	6.25	12	5	68	28.33	69	28.75	2	0.84	8	3.33	240	100

Figure 4 explained that the prospective teacher's understanding of the concepts in constructing proof was unrevealed, especially on his concept images. The language required to express mathematical ideas in the proof was still insufficient and unclear. He began to construct a proof by letting an odd number as *a* and then being manipulated in an equation resulting a=1. However, it was not clear to bring the proof into corresponding directions of a valid proof that was *the sum of two odd numbers is an even number*. Based on the data, 2.5% of the 240 prospective teachers experienced such proving errors.



Translation

Prove that the sum of two odd numbers is an even number.

Proof:

Suppose *n* is an arbitrary number, then (2n-1) is an odd number.

n(1) = 2(1) – 1 = 1

n(2) = 2(2) – 1 = 3

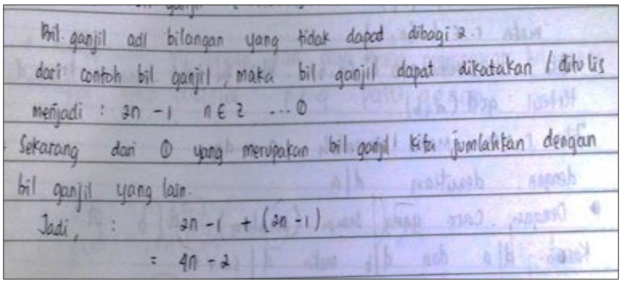
n(3) = 2(6) – 1 = 11

(2n – 1) + (2n – 1) = 4n – 2

For n ∈ Z then n(1) = 4(1) – 2 = 2; n(2) = 4(2) – 2 = 6; n(3) = 4(3) – 2 = 10 (integer).

Figure 5. Example of Proof AD4

Figure 5 shows that the prospective teacher started to use functional language that was supposed as *n*, but it was not completed. The results show that the prospective teacher used several examples to understand the concept of constructing a proof. Moreover, he was still unable to build their examples, which were used to construct proofs. Based on the data, 2.08% of the 240 prospective teachers experienced such proving errors.



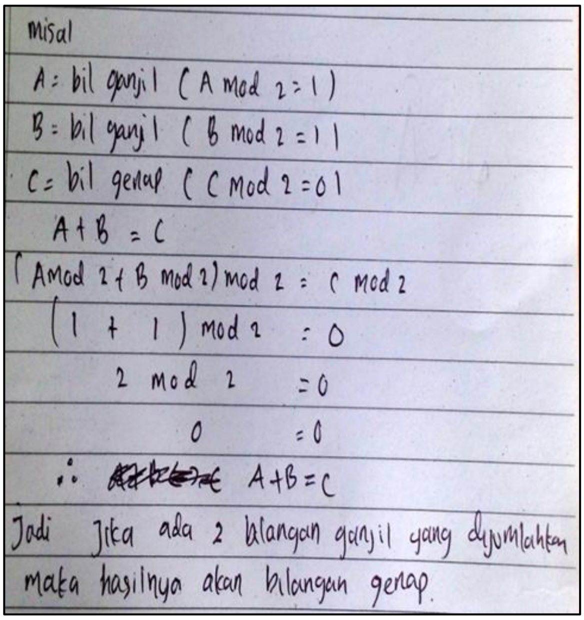
Translation

Odd number is a number that cannot be divided by 2. Based on this definition, then an odd number can be said or written as 2n-1, n ∈ Z …(1)

From (1), we will add with the order odd number. Thus, 2n-1 + (2n-1) = 4n-2.

Figure 6. Example of Proof AD5

Figure 6 shows that the prospective teacher could involve deductive reasoning with a definition, but she still did not know how to use the definition of an odd number correctly. This was indicated by the use of a definition of an odd number namely 2n – 1. Afterwards, she added the other odd number that resulted (2n - 1) + (2n - 1). Based on the sum of these numbers, she used the same variables namely n, however, letting two arbitrary odd numbers with the same variables was not accepted due to the possibility that those two numbers could be different. Based on the data, 27.5% of the total prospective teachers experienced such proving errors.



<u>Translation</u>

Suppose

A = odd number (A mod 2 = 1)

B = odd number (B mod 2 = 1)

C = even number (C mod 2 = 0)

A + B = C

(A mod 2 + B mod 2) mod 2 = C mod 2

(1 + 1) mod 2 = 0

2 mod 2 = 0

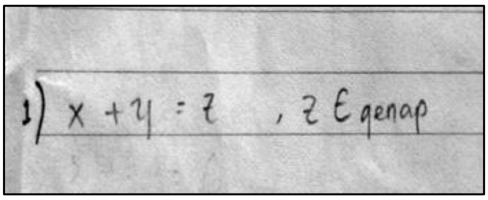
0 = 0

Thus, A + B = C

We can conclude that the sum of two odd numbers is an even number.

Figure 7. Example of Proof AD6

Figure 7 explains that deductive reasoning was correctly used by the prospective teacher's proof. However, the prospective teacher used particular language and mathematical notation incorrectly. This problem could be seen from the notation "=", which meant "is equal" instead of "is equivalent". The symbol was normally used in congruence involving modulo. Therefore, it could be concluded that the prospective teacher did not fully understand the meaning of a notation "=". Based on the data, 0.42% of the total prospective teachers experienced such proving errors.



<u>Translation</u>

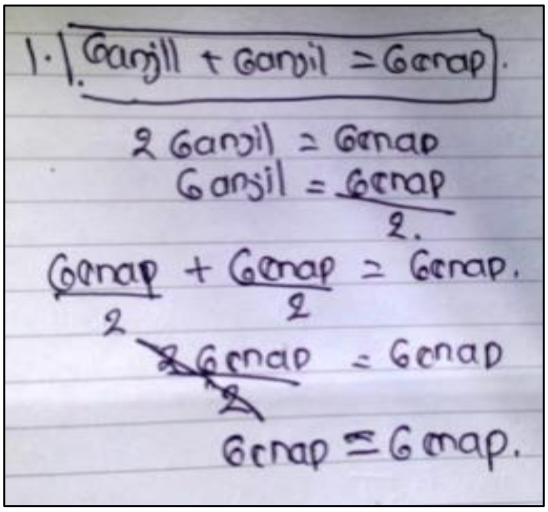
x + y = Z, Z ∈ even number.

Figure 8. Example of Proof AD7

Figure 8 shows that the prospective teacher involved deductive reasoning in constructing the proof. The finding x + y = z showed that the prospective teacher used functional language at the beginning of the proof. However, this equation was meaningless due to the poor mathematical argument and interpretation of the symbols given. The prospective teacher in this proof likely had less knowledge of proving so it was difficult to start constructing a proof. Based on the data, 3.33% of the total prospective teachers experienced such typical proving errors.

Proof B

In Proof B, the prospective teachers involved deductive reasoning and other languages, drawings, and movable objects during constructing a proof. In this category, 0.84% of the prospective teachers had difficulties in constructing a proof caused by several things, comprising a less understanding of the concept (0.42%) and less understanding of mathematical notations and language in constructing a proof (0.42%). The following examples show the results of prospective teachers' answers that contained errors in constructing Proof B.



Translation

Odd + odd = even

2 Odd = even

Odd $= \frac{\text{even}}{2}$

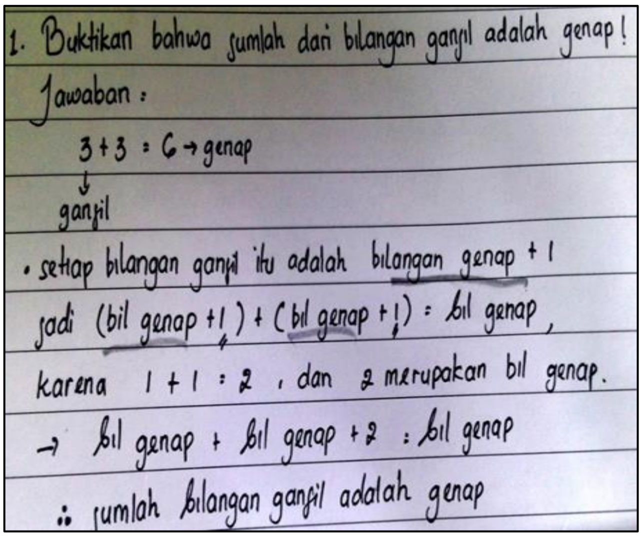
$\frac{\text{even}}{2} + \frac{\text{even}}{2} = \text{even}$

$\frac{2\ \text{even}}{2} = \text{even}$

even = even

Figure 9. Example of Proof BD1

Figure 9 explains that the prospective teacher involved deductive reasoning but did not use functional language in constructing a proof. It could be seen from his work showing that "odd + odd = even". He also could not state the definition correctly as indicated in his definition that "odd = $\frac{even}{2}$". Whereas, when an even number was divided by 2, the result was also an even number. Hence, he performed the proof incorrectly. Based on the data, 0.42% of the total prospective teachers experienced such proving errors.



Translation

Prove that the sum of two odd numbers is an even number.

Answer:

3 + 3 = 6, where 3 is an odd number, and 6 is an even number.

Every odd number is even number + 1. Thus, (even number + 1) + (even number + 1) = even number, cause 1 + 1 = 2, and 2 is even number.

→ Even number + even number + 2 = even number.

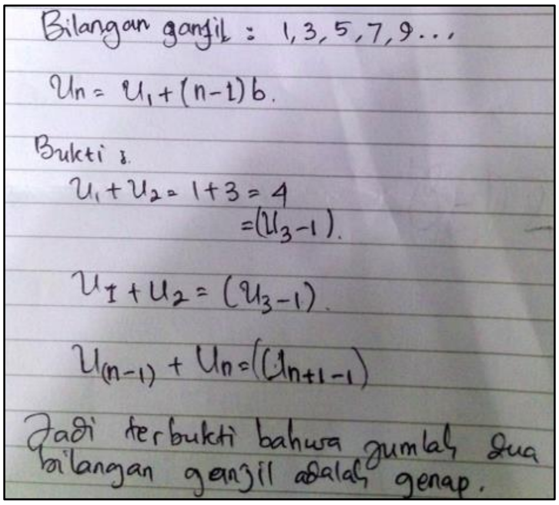
Thus, the sum of odd numbers is an even number.

Figure 10. Example of Proof BD6

In accordance with Figure 10, it was indicated that the prospective teacher already involved deductive reasoning but she demonstrated the proof by her language without the use of appropriate functional language in constructing a proof. This could be seen from her sentence *every odd number is an even number plus one*. This sentence should employ some symbols using a functional language, for example, 2n + 1 for an odd number with n integer. Thus, the prospective teacher still did not understand how to use the symbolic language in proof. It could be a result of the limitations of her conceptual understanding about the nature of proof. Based on the data, 0.42% of the total prospective teachers experienced such typical proving errors.

Proof C

In this type of proof, the prospective teachers were not able to prove using inductive reasoning and other languages, drawings, and movable objects. However, 31.25% of the prospective teachers got difficulties in constructing this type of proof caused by less understanding of mathematical concepts. The following example shows student's answer that had difficulties in constructing Proof C.


Bilangan ganjil : 1, 3, 5, 7, 9...
$U_n = U_1 + (n-1)b$.
Bukti :
$U_1 + U_2 = 1 + 3 = 4$
$= (U_3 - 1)$.
$U_1 + U_2 = (U_3 - 1)$.
$U_{(n-1)} + U_n = (U_{n+1-1})$
Jadi terbukti bahwa jumlah dua bilangan ganjil adalah genap.

Translation

Odd number = 1, 3, 5, 7, 9,…

$U_n = U_1 + (n-1)b$

Proof:

$U_1 + U_2 = 1 + 3 = 4$

$= (U_3 – 1)$

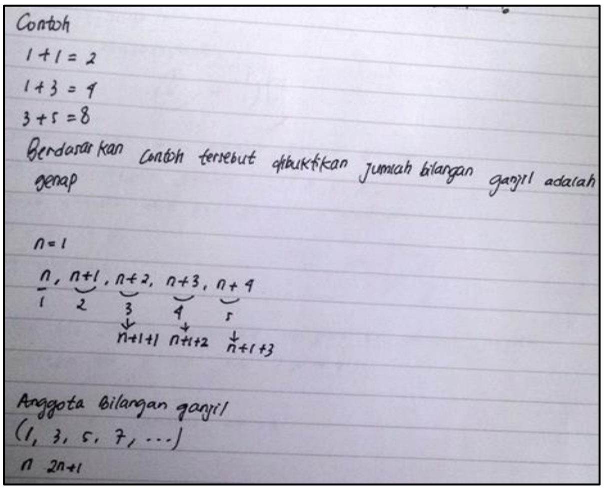
$U_1 + U_2 = (U_3 – 1)$

$U_{(n-1)} + U_n = (U_{n+1-1)}$

Thus, it is proven that the sum of two odd numbers is an even number.

Figure 11. Example of Proof CD2

Figure 11 shows that the prospective teacher tried to perform inductive reasoning by giving some examples of the number involved in an arithmetic equation at the beginning of stating a proof. Nevertheless, it was unclear that the concept of proof used in constructing a proof was well presented. For example, from the equation U1 + U2 = 1 + 3 = 4 = (U3- 1), the prospective teacher concluded that U(n-1) + U2 = (Un + 1-1). In this case, the prospective teacher still did not understand the whole direction of the proof due to the lack of an intuitive understanding about how a mathematical proof should work. Therefore, he could not finish constructing the proof correctly. Based on the data, 2.5% of the total prospective teachers experienced such typical proving errors.



Translation

1 + 1 = 2   1 + 3 = 4   3 + 5 = 8

Based on this example, it is proven that the sum of two odd numbers is an even number.

n = 1

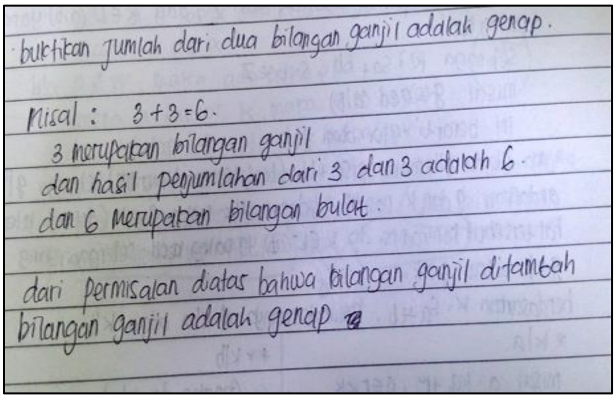
n→1, n+1 → 2, n+2→3→n+1+1, n+3→4→n+1+2, n+4→5→n+1=3

The members of odd number: (1, 3, 5, 7,…)

n 2n+1

Figure 12. Example of Proof CD3

Figure 12 depicts the prospective teacher performed inductive reasoning by giving an example of numbers. However, it was noted that her understanding of the concept of proof was still unclear. The prospective teacher started from stating the proof by giving some examples, then generalized them into the formal form. Afterwards, she continued to resume her work by providing other examples. It was incompatible with the concept of inductive proof where the valid examples satisfying the condition of a statement should be generalized into a formal conclusion. Based on the data, 2.5% of the total prospective teachers experienced such typical proving errors.



Translation

Prove that the sum of two odd numbers is an even number.

Suppose: 3 + 3 = 6

Where, 3 is an odd number and the result of 3 + 3 = 6, 6 is an integer.

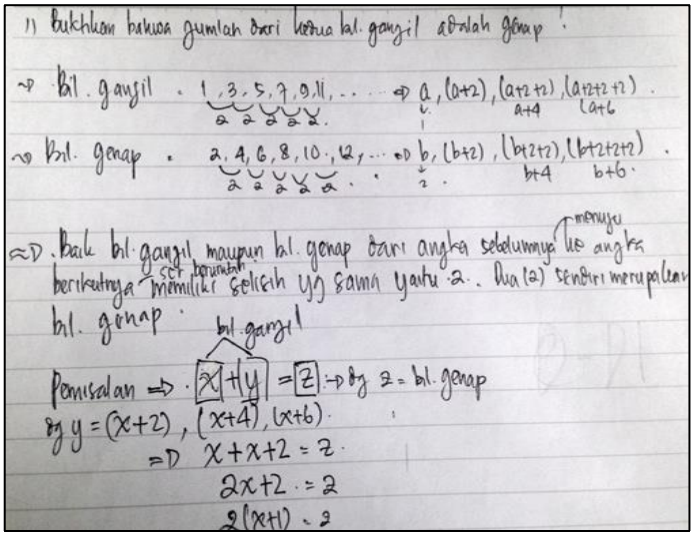
Therefore, the sum of two odd numbers is an even number.

Figure 13. Example of Proof CD4

Figure 13 portrays that the prospective teacher used examples to construct an inductive proof but she did not complete the proof. She was more inclined to mention some numbers on an arithmetical operation, namely 3 + 3 = 6, but did not conclude her examples of proving to the general form, which was a functional language used in constructing a proof. Based on the data, 26. 25 % of the total prospective teachers experienced such proving errors.

Proof D

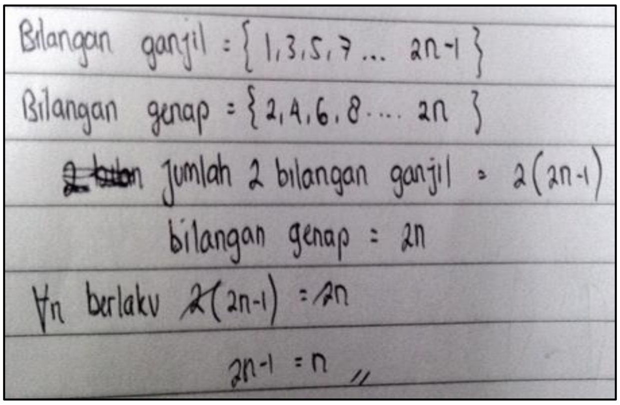
In relation with Proof D, the prospective teachers could not perform the proving process correctly based on inductive reasoning and functional language in constructing a proof. The number of prospective teachers who got proving difficulties in this category was 6.25%, in which the problem was caused by a weak understanding of the concept in the proof task. The following examples show the prospective teacher's answers that had difficulties in constructing Proof D.



Translation
Prove that the sum of two odd numbers is an even number.
Odd numbers = 1, 3, 5, 7, 9, 11, … → = a, (a + 2), (a+2+2), (a+2+2+2) = a, a + 2, a +4, a + 6
Even numbers = 2, 4, 6, 8, 10, 12, …
→ = b, (b + 2), (b+2+2), (b+2+2+2)
= b, b + 2, b + 4, b + 6
For either odd or even numbers, the difference between every two consecutive number is 2. And, 2 is an even number. Where x, y are odd numbers and z is an even number.
With y = (x+2), (x+4), (x+6)
x + x + 2 = z; 2x + 2 = z; 2(x+1) = z

Figure 14. Example of Proof DD1

Figure 14 shows that the prospective teacher did not understand how to define an odd number in the form of functional language. It could be seen from the student's answer on the definition of even and odd numbers. The prospective teacher gave a series of example on how odd numbers and even numbers are illustrated as evidence that he started proving inductively. Despite he tried to arrange the general form of the examples, which was the series a, (a + 2), (a+2+2), (a+2+2+2) for even number, and b, (b + 2), (b+2+2), (b+2+2+2) for odd number. However, it did not show how an odd number and an even number should be mathematically symbolized. Thus, his final step, which was 2(x+1) = z yielded a variety of interpretations, did not certainly describe the condition expected in the proof task. Based on the data, 5 % of the total prospective teachers experienced such proving errors.



Translation

Odd numbers = {1, 3, 5, 7, … 2n – 1}

Even numbers = {2, 4, 6, 8, … 2n}

The sum of two odd numbers = 2(2n – 1)

Even numbers = 2n

∀ n apply 2(2n – 1) = 2n

2n – 1 = n

Figure 15. Example of Proof DD5

Figure 15 portrays the prospective teacher used inductive reasoning but he still did not know how to use the definition correctly. The figure also shows (2n-1) = n that indicated that he did not use the definition of odd number correctly. He canceled number 2 on the right and left instead of using the definition of an odd and even number that he had written. Based on the data, 1.25% of the total prospective teachers experienced such proving errors.

Our finding indicates that the prospective teachers apply deductive and inductive methods in constructing a proof. The deductive method consists of two types namely Proof A and Proof B and the inductive method consists of two types covering Proof C and Proof D. Table 4 points out that 62.5% of the prospective teachers use deductive method while 37.5% of them use inductive method. Meaning that, more than half of the prospective teachers' answers use deductive method. Despite some of the prospective teachers having errors in constructing a proof, they already try to construct a proof with deductive and inductive methods. This finding is consistent with the research conducted by Miyazaki (2000) that most students in his study use a deductive method instead of the inductive one in constructing a proof.

Furthermore, the prospective teachers perform Proof A, Proof B, Proof C, and Proof D types with the percentage of 61.66%, 0.84%, 31.25%, and 6.25%, respectively. Therefore, it shows that Proof A is the most commonly found in the prospective teachers' answers than those of other types. When compared to the findings undertaken by Kögce et al. (2010), it does not align with the results of Kögce et al. (2010), in which the study result reports that Proof C is performed by most students than the other types of proof (51.2%). The fact that our study has found many deductive methods in our participants' answers might occur because the proof task given in the present study demands a solver to use a deductive method instead of an inductive one. However, in connection with the results of the prospective teacher's answers, there are some answers indicating an inductive method, all of which are still incorrect. This finding very likely corresponds to Demiray and Bostan (2017) who report that most of the students' incomplete proof yielded incorrect proofs are caused by the unsuitable inductive method the students use in constructing their proofs. Therefore, it indicates that the type of proof task affects the selection of proving methods.

The results of the present study are also consistent with Miyazaki's (2000) study, where Proof A is performed by the highest number of prospective teachers. The results show that prospective teachers can use deductive reasoning and sufficient techniques of proof, however, there are still some errors in constructing a proof. In addition, Miyazaki (2000) points out that Proof C is performed by the least number of prospective teachers. This does not align with the present study that reveals the fact that Proof C is performed by the second-highest number of prospective teachers. This result shows that the prospective teachers still involve inductive reasoning with other languages, drawings, and objects used in the process of constructing a proof. In this regard, promoting deductive argumentation among students in mathematics education is important since many prospective teachers, including those who enrolled postgraduate study program in mathematics education, often perform logically disconnected premises and conclusions drawn within a mathematical proof (Ndemo, 2019).

Regarding the difficulties in proving, the results show that AD5 type is performed by 27.5% of the prospective teachers, which becomes the highest result of proof. Similarly, Edwards and Ward (2004) convey that many prospective teachers cannot use the definition to make mathematical proofs. Aligned with Moore's category, the use of definition is indeed being one of the difficulties in constructing a mathematical proof. In addition, the present study shows that the number of prospective teachers performing incorrect proofs is bigger than those who make the correct ones. That is, most prospective teachers still experience difficulties in constructing a mathematical proof. 79.98% of the prospective teachers are still weak in understanding the concept of proof, 3.33% of them lack of knowledge, and 0.84% of them get limited ability to construct a proof. This study is consistent with Chin and Lin's (2009) study revealing that most prospective teachers have problems in constructing valid algebraic proofs.

The present study not only indicate the performance of prospective teachers regarding types of reasoning and difficulties in proving processes, but also show the potential use of analytical framework of assessing individuals' proving performance. This framework is developed in order to get a broader insight on how to evaluate types of proof and proving difficulties from a written response representing individual proof performance. It is expected that this analytical framework will complement other frameworks in assessing individual cognitive performance related to proof, such as mathematical knowledge for teaching proof (Buchbinder & McCrone, 2020) as the knowledge of different types of proofs becomes one of vital components in this framework. However, the combined Miyazaki-Moore method used in the framework is carried out to only assess students' written responses. In future, it is more beneficial if the data include interview results to confirm the detailed information about the students' difficulties in constructing proofs.

CONCLUSION

The most prospective teachers construct a proof by employing deductive reasoning rather than inductive reasoning. It can be seen in Proof A that has been performed by the highest number of prospective teachers. However, some prospective teachers still experience difficulties in constructing a mathematical proof. The types of difficulties mostly found in the prospective teachers' answers include the fact that they cannot appropriately use the definition in making mathematical proofs.

This study only presents the prospective teachers' responses in constructing a proof because the researchers want to know the trend of mapping models in assessing prospective teachers regarding their knowledge about proof constructions. They also have empirically proven that the framework proposed in this study can work. The advantages of using the framework cover the ability to assess students' types of reasoning and difficulties in constructing proofs simultaneously.

As the constructive feedbacks, the framework can be used as an evaluation tool for the needs of mathematics teacher education program in a university curriculum. For further studies, the present study offers a potentially broader insight on assessing learners' cognitive processes to study learners' reasoning process in mathematical proof regarding proving difficulties and types of reasoning since the framework developed in this study has not covered such issue yet. In addition, this framework is intended to only code the responses based on the participants, meaning that every single response can only get chance to be coded in one category of proof based on types of reasoning and difficulties. Thus, it is suggested that the framework can be developed into covering more than one category of proof since, for example, a response from another mathematical proof task may be categorized in more than one type of difficulties.

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