Journal on Mathematics Education
Volume 16, No. 3, 2025, pp. 937-954

Mathematical reasoning: How students learn mathematics?

Sutarto Hadi[1,*], Maarten Dolk[2], Kamaliyah[1], Taufiq Hidayanto[1]

[1]Department of Mathematics Education, Universitas Lambung Mangkurat, Banjarmasin, Indonesia
[2]Freudenthal Institute, Utrecht University, Utrecht, Netherlands
*Correspondence: sutarto.hadi@ulm.ac.id

Received: 18 June 2025 | Revised: 13 August 2025 | Accepted: 20 August 2025 | Published Online: 25 August 2025

Abstract

Mathematics learning is widely recognized as a fundamental component of school curricula, as it equips students with essential competencies, particularly mathematical reasoning, which underpins logical analysis, problem solving, and decision making. The importance of cultivating reasoning skills is especially pronounced in the current era of disruption, characterized by rapid advances in information and communication technology and the automation of human labor by machines and autonomous systems. As physical tasks are increasingly performed by technology, human capacities such as reasoning and emotional intelligence become critical. Mathematical reasoning provides the foundation for understanding concepts, formulating logical arguments, and generating solutions across domains such as the natural sciences, society, and engineering, while also enabling students to approach problems critically and systematically. However, despite its significance, research in primary education has often emphasized procedural knowledge rather than examining how students construct and apply reasoning when confronted with mathematical challenges, leaving a gap in understanding how reasoning develops in authentic classroom contexts. To address this issue, the present study investigates how Grade 4 and Grade 5 students in a primary school in Banjarmasin, Indonesia, employ mathematical reasoning strategies to solve non-routine problems. Through a classroom-based experimental approach, we analyzed students' solution pathways and the reasoning patterns they demonstrated in navigating mathematical tasks. The findings offer insights into the developmental characteristics of mathematical reasoning in upper primary school and contribute to broader discussions on fostering reasoning skills effectively, with implications for designing mathematics instruction that prepares students to meet the cognitive demands of an era increasingly shaped by automation and technological disruption.

Keywords: Artificial Intelligence, Information and Communication Technology, Mathematical Reasoning, Students Learning

How to Cite: Hadi, S., Dolk, M., Kamaliyah, & Hidayanto, T. (2025). Mathematical reasoning: How students learn mathematics?. *Journal on Mathematics Education, 16*(3), 937–954. https://doi.org/10.22342/jme.v16i3.pp937-954

Many consider May 11, 1997, as the symbolic beginning of humanity's challenge against machines, when World Chess Champion Garry Kasparov was defeated by IBM's Deep Blue computer. The development of technologies enabling machines to think and learn—extending beyond robotics—has been a persistent human endeavor (Prabhu, & Premraj, 2025). For instance, Deep Blue could evaluate between 100 and 200 billion positions in just three minutes, demonstrating computational capabilities far beyond human capacity (Hsu, 2022). Since Kasparov's defeat over two decades ago, artificial intelligence (AI) has advanced at an unprecedented pace, raising concerns that excessive reliance on AI could diminish human agency.

AI is now embedded in everyday life, most notably through smartphones, which serve as ubiquitous interfaces for intelligent systems (Lee et al., 2023). The current era is characterized by profound disruptions driven by advancements in information and communication technologies (Carayannis et al., 2022). Many aspects of human labor are increasingly being replaced by machines, leaving humans primarily with cognitive and emotional capacities when autonomous systems assume routine tasks.

A strong foundation in mathematical reasoning is essential for understanding mathematical concepts and solving problems in a wide range of real-world contexts (Sun et al., 2025). Such reasoning connects mathematics to engineering, social sciences, and the natural sciences. Mathematical reasoning provides a framework for logical thinking, problem solving, and decision making (Herbert & Williams, 2023). It enables students to critically analyze problems and formulate appropriate solutions. Furthermore, Ernie et al. (2023) emphasizes that mathematical reasoning allows individuals to move beyond rote memorization of facts, rules, and procedures; it fosters the ability to make conjectures based on prior experience, thereby facilitating a deeper and more meaningful understanding of interconnected mathematical concepts. Reasoning, both inductive and deductive, is central to learning mathematics, as it cultivates students' logical thinking and problem-solving abilities.

The National Council of Teachers of Mathematics (NCTM, 2000) highlights the critical role of mathematical reasoning, advocating for its exploration in every classroom, across all students, and in connection with all mathematical content. Mathematical processes should be integrated into instruction from the earliest grades, enabling students to engage in problem solving, problem posing, explanation of thinking, and evidence-based reasoning (Torres-Peña et al., 2025). Teachers are encouraged to connect mathematical concepts, relate mathematics to other disciplines, and support multiple representations of the same mathematical situations. Finally, Baroody and Coslick (1998) identify several benefits of training students in mathematical reasoning:

1. Students gain direct experience in observing patterns, formulating conjectures, and evaluating them, which deepens their understanding of mathematical processes.
2. Students develop confidence in making conjectures, even when uncertain of exact answers, reducing anxiety associated with problem-solving.
3. Students learn the value of negative feedback in refining their reasoning and decision-making processes.
4. Students recognize the importance of intuition, inductive and conjectural reasoning, and deductive proof, understanding that intuition underpins higher-order thinking in mathematics and other scientific domains.

Despite its importance, mathematics education in schools does not always successfully cultivate students' reasoning skills (Mukuka et al., 2023; Ramlan et al., 2025). One contributing factor is teacher preparedness. A survey conducted by the U.S. Department of Education reported that only 63% of approximately 144,000 high school mathematics teachers held both a mathematics major and certification, 26% held either a major or certification, and 11% held neither (Sousa, 2015). The situation in Indonesia may be comparable. Teachers are therefore a pivotal determinant of the effectiveness of mathematics education. Ideally, teachers should facilitate the development of students' reasoning skills; however, current attention to this goal remains limited. Pramudiani (2023) identifies several factors necessary for teachers to effectively foster mathematical reasoning:

1. Knowledge of learning trajectories and pathways
2. Understanding of how students learn mathematics
3. Ability to accurately observe and interpret student behavior during instruction
4. Competence in applying goal-oriented and diagnostic teaching methods
5. Awareness of diverse student characteristics

Framework for Mathematical Reasoning

One of the seven elements of mathematical intelligence that humans possess is mathematical reasoning, as highlighted by Junaid (2022) in Mathematical Intelligence: A Story of Human Superiority over Machines—What Humans Have That Robots Don't. Junaid (2022) argues that mathematics offers the most powerful logical framework for establishing timeless truths, emphasizing that our ability to reason protects us from accepting questionable assertions produced by systems reliant solely on pattern recognition. The rapid rise of large-scale artificial intelligence (AI) technologies may have shaped his perspective. In an era where automation and "superintelligent" machines pose potential threats, Junaid (2022) raises a critical question: what does it mean to be human? He frames this as a profound challenge to our way of living and thinking.

Reasoning and mathematics are deeply interconnected (Callingham & Siemon, 2021). To strengthen their mathematical understanding, all children should be provided with opportunities to engage in mathematical reasoning (Mukuka et al., 2023). Regular exposure to mathematical tasks helps students internalize mathematical thinking until it becomes a habitual cognitive process (Fonseca, 2018). Furthermore, Bragg et al. (2015) contend that students should be challenged to solve problems that develop their capacity for reasoning from an early age, as reasoning is essential for forming conclusions. Because reasoning forms the cornerstone of a deep understanding of mathematics, it must be embedded consistently in students' mathematical experiences beginning as early as kindergarten.

Fonseca (2018), citing the National Council of Teachers of Mathematics (NCTM, 2000), asserts that students should be able to "recognize reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs; [and] select and use various types of reasoning and methods of proof" as a result of mathematics education. NCTM (2000) further advocates the use of meaningful tasks and open-ended questions to foster reasoning and problem solving, highlighting that effective mathematics instruction should allow multiple entry points and diverse solution strategies, encouraging students to solve, discuss, and reflect on mathematical tasks.

Bragg et al. (2015) define mathematical reasoning as the process by which students clarify ideas, draw inferences, transfer knowledge across contexts, assess the validity of claims, compare and contrast ideas, and justify their conclusions. Finally, mathematical reasoning can be classified into four categories:

1. Inductive reasoning – A process that uses analogies, examples, observations, and experiences to make generalizations. Inductive reasoning derives general conclusions from specific cases, often by recognizing patterns, filling conceptual gaps, and explaining why certain counterexamples should be rejected.
2. Deductive reasoning – A process fundamental to mathematics and logic in which conclusions necessarily follow from given premises, ensuring that if the premises are true, the conclusion cannot be false.
3. Abductive reasoning – A process of generating plausible hypotheses or explanations that account for observed phenomena, often resulting in novel insights or predictions.
4. Adaptive reasoning – One of the five strands of mathematical proficiency, along with conceptual understanding, procedural fluency, strategic competence, and productive disposition (Kilpatrick et al., 2001). Adaptive reasoning encompasses inductive, deductive, and abductive reasoning and is reflected in students' abilities to justify, explain, and reflect on their mathematical practices. Students engage in adaptive reasoning when they use facts, procedures, and concepts to construct and evaluate solutions, generalize findings, and refute conjectures through counterexamples (Bragg et al., 2015).

METHODS

We conducted a classroom-based qualitative study in a partner primary school to investigate the processes of teaching and learning division using a realistic mathematics education (RME) approach. Prior to classroom implementation, we collaborated with the classroom teachers to design context-based mathematical problems aligned with the topics currently taught in their classes.

Grade 4 Intervention

At the time of the study, the Grade 4 class was learning the concept of division. Together with Hamidah, an experienced teacher with more than ten years of teaching experience, we designed a lesson around a contextual problem involving division with whole numbers. The problem posed to students was as follows:

> ***Uncle Gathering Dragon Fruit***
> *Uncle collected 98 dragon fruits from his garden. He wishes to distribute them equally among his relatives, both near and distant. To do so, he will pack the dragon fruits into boxes, with six fruits per box. How many boxes are needed?*

The class consisted of 32 students. We discussed the scenario with the teacher before implementing the lesson. The instructional design followed a four-step didactical intervention:

1. Problem Presentation – The teacher presented the contextual problem and instructed students to solve it individually for approximately ten minutes.
2. Group Work – Students were then divided into groups of four to compare and discuss their solutions.
3. Poster Production – Each group summarized their solution strategy on a poster for presentation.
4. Whole-Class Discussion – Groups presented their posters to the class, explained their reasoning, and received feedback from peers and the teacher.

This structure aimed to stimulate individual reflection, peer interaction, and collective knowledge construction (see Figure 1 for problem presentation).

Figure 1. Problem presentation

Grade 5 Intervention

A similar intervention was conducted in a Grade 5 class of 36 students, led by Yusri, a mathematics teacher with over 20 years of experience. At the time, the class was studying division involving fractions. Together with Yusri, we co-designed a problem that was both contextually meaningful and mathematically relevant to the topic as shown in Figure 2.

Rice Consumption Problem

Father bought a 25-kg bag of rice.
Each day, Mother cooks $\frac{3}{4}$ kg of rice.
For how many days will the rice last?

Figure 2. Contextual problem for division by fraction

The same four-step didactical intervention (problem presentation, individual work, group discussion, and whole-class discussion) was implemented in this lesson (see Figure 2).

Data Collection and Analysis

Given the exploratory nature of the study, a qualitative research approach was employed. Data were collected through multiple sources, including students' written solutions, participant observation, planned and spontaneous conversations, audio and video recordings, and field notes. Triangulation of data sources was used to enhance the trustworthiness of findings.

The partner school where the study was conducted is part of our ongoing collaboration to develop and implement RME-based instructional practices. The RME approach emphasizes contextual problems as starting points for learning mathematics, fostering student engagement and sense-making. Interactivity is a key principle of the approach: students are encouraged to explore problems using their own strategies and representations before formalizing their mathematical understanding.

RESULTS AND DISCUSSION

Our findings confirm that providing students with opportunities to reason informally before introducing formal methods can facilitate deeper understanding. This aligns with RME principles, which view students' own strategies and representations as crucial steppingstones toward formal mathematical reasoning (Rianasari & Guzon, 2024; Siswantari et al., 2025).

Grade 4 Intervention

In the Grade 4 classroom, the lesson did not unfold entirely as anticipated. The primary challenge was that students were not accustomed to verbalizing their thinking or explaining their reasoning processes. Although they were capable of arriving at correct solutions, they often struggled to articulate the steps they had taken. This presented a challenge for the teacher, who found it difficult to support students in expressing their ideas effectively. The teacher further explained that this difficulty was partly due to the prolonged school closures during the COVID-19 pandemic, during which students had been learning from home for more than two years. This disrupted their opportunities for classroom interaction and mathematical discourse, which are essential for the development of reasoning skills.

Grade 5 Intervention

In contrast, the Grade 5 classroom produced richer discussions and more interactive participation. A particularly notable moment occurred when a student volunteered to present his solution to the class. His explanation sparked dialogue among peers, and several students shared alternative strategies for solving the problem. We also collected samples of students' written work, which offered valuable insights into their reasoning processes.

The students' solutions illustrated a form of horizontal mathematization, a central concept in RME. Rather than immediately applying formal algorithms, students constructed their own solution methods, using personal symbols and informal representations that made sense to them. This process reflects a transition from informal to formal mathematical understanding and demonstrates the importance of allowing students to reason through problems in ways that align with their cognitive development.

Formal and Informal Knowledge in Students' Reasoning

The analysis in this study focuses on students' mathematical reasoning as expressed through both formal and informal knowledge. Scholars differ on how to delineate these two forms of knowledge. For example, Ginsburg et al. (2001) classify counting as informal knowledge, whereas Resnick (1989) considers it formal knowledge. Furthermore, Chiu and Tron (2004) provide a useful distinction:

1. Informal knowledge refers to knowledge acquired through children's interaction with the physical and social world, often involving non-numerical quantities or non-standard symbols. For example, children may compare sets without counting exact quantities.
2. Formal knowledge refers to knowledge that involves systematic manipulation of symbol systems (e.g., writing numbers, using symbolic patterns such as AB–AB, or applying mathematical operations). Formal knowledge is typically acquired through structured classroom instruction.

Chiu and Tron (2004) argue that children frequently develop informal mathematical knowledge before they are able to verbalize it explicitly. Resnick (1989) categorizes children's non-numerical quantitative knowledge into four types:

1. Absolute quantity judgment – using labels such as big, small, many, or few to describe size or amount.
2. Comparative language – using relational terms such as bigger, smaller, taller, or shorter to compare quantities.
3. Change in quantity – recognizing increases or decreases in sets (e.g., if one cookie is taken away, one fewer remains).
4. Part–whole schema – understanding that a subset is smaller than the whole (e.g., a slice of bread compared to a full loaf).

Prior research has established that children's informal mathematical knowledge provides an essential foundation for the acquisition of formal mathematical knowledge (Papandreou & Tsiouli, 2022; Panaoura & Nitsiou, 2023; Xu & Cai, 2024). For example, children often use informal comparison strategies and create mental number lines to determine which number is larger (Resnick, 1989). They also apply intuitive concepts of change to perform informal addition and subtraction on sets (Ginsburg, 2001). When children enter formal schooling, this informal knowledge can be enriched and formalized, enabling them to work with abstract mathematical objects and symbolic systems (Chiu & Tron, 2004).

Analysis of Students' Works on Dragon Fruit Problem

The majority of students demonstrated an understanding of the problem posed by the teacher. Their typical approach began with drawing a square to represent a box and then filling it with six dragon fruits, as indicated in the problem statement. Students constructed boxes sequentially, each containing six dragon fruits, until all 98 dragon fruits were accounted for. The remaining fruits were placed in a final, partially filled box. While individual solutions varied slightly, common strategies relied on drawing, logical reasoning, and incremental counting. These strategies can be classified into four main models:

Solution 1: Concrete Representation through Drawing

In the most common solution approach as shown in Figure 3, students drew a square for each box and manually filled each square with six individual dragon fruits. These included the following: (1) creating a box and filling it with six dragons-fruits; (2) creating a box and filling it with six dragons, in this case six dragon-fruits are represented by the number 6; (3) creating a box and filling it with the number 6, then repeating addition; (4) creating a box and filling it with the number 6, then counting jumping 6 as on a number line, i.e. 6 + 6 = 12, 12 + 6 = 18, 18 + 6 = 24, and so forth. This highly concrete approach provided students with a visual and tactile method for tracking quantities. However, it also led to a tedious and time-consuming process, as students had to count sequentially from 1 to 96 to verify the total number of fruits placed in the boxes. Despite its inefficiency, this strategy appeared to give students confidence in their results. Nevertheless, several students miscounted the number of remaining dragon fruits, leading to incorrect conclusions about the number of boxes required.

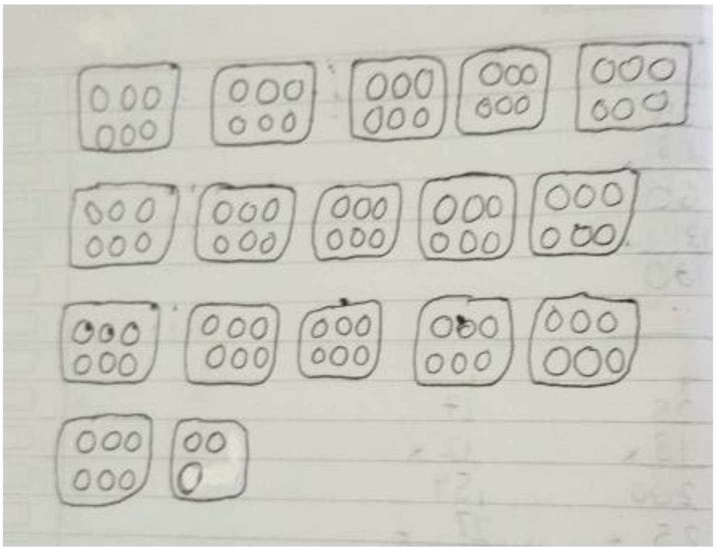

Figure 3. Creating a box and filling it with six dragon fruits

Solution 2: Symbolic Representation of Quantity

Some students adopted a more efficient approach illustrated in Figure 4 by replacing the individual drawings of dragon fruits with the numeral 6, thereby symbolically representing the quantity in each box. This transition to symbolic representation reflects a shift toward abstraction. Nevertheless, students still faced the challenge of tracking cumulative totals and often repeated their work on a second page when space was insufficient.

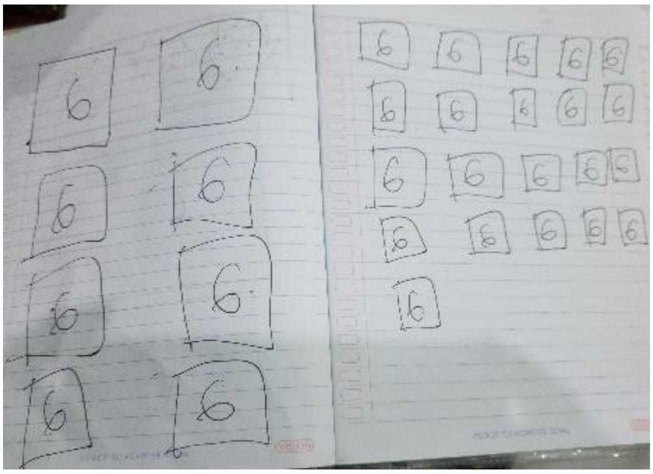

Figure 4. Creating a box and filling it with six dragons, in this case six dragons are represented by symbol of 6

Although slightly more efficient than Solution 1, the process remained labor-intensive, and students relied on mental counting or fingerspelling to confirm their results as shown in Figure 5.

Figure 5. Counting on fingers for justification of answer

Solution 3: Repeated Addition

A subset of students employed repeated addition as a means of tracking quantities presented in Figure 6. Rather than recounting from the beginning for each box, students wrote cumulative sums above each box, such as $6 + 6 + 6 + \cdots + 6 = 96$. This strategy reflects a higher level of reasoning than the first two approaches, as students demonstrated an understanding of grouping and accumulation. They correctly identified that 16 boxes contained 96 dragon fruits, and that the remaining 2 fruits would be placed in a 17th box.

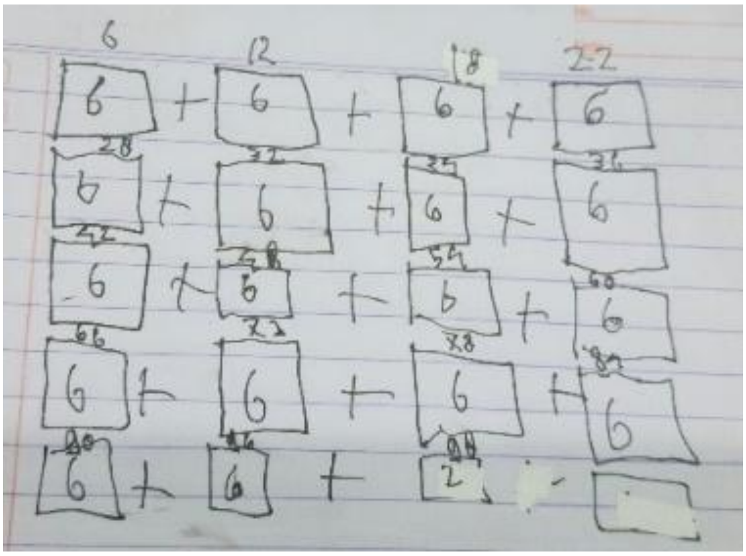

Figure 6. Creating a box and filling it with the number 6, and repeating addition

Solution 4: Jump Counting and Number Line Representation

Another group of students extended Solution 3 by using a number-line representation to "jump" by sixes (e.g., 6, 12, 18, 24, …) illustrated in Figure 7. This approach made the process of repeated addition more visual and systematic, reducing cognitive load and facilitating verification of intermediate results. The use of number lines is pedagogically significant because it bridges the gap between concrete counting strategies and more abstract symbolic reasoning.

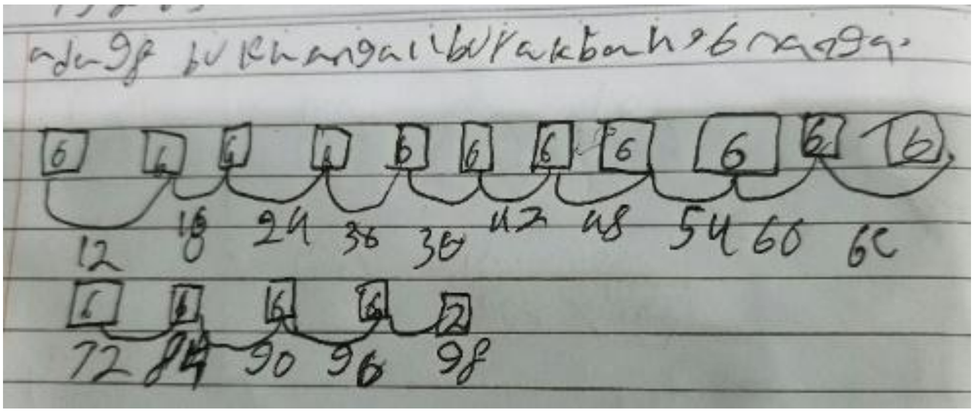

Figure 7. Make a box and fill it with 6 then count the jumps of 6

Emergence of Multiplicative Reasoning

Notably, at least one student demonstrated a transition from repeated addition to multiplicative reasoning as shown in Figure 8. This student represented the situation using boxes, filled each with six dragon fruits, and then confirmed the result using the equation 6 × 16 = 96. After determining that two fruits remained, the student placed them in a final box, thus verifying that 17 boxes were needed in total. This solution illustrates the student's ability to formalize reasoning by connecting a concrete representation with an abstract mathematical operation, a key step in the process of vertical mathematization.

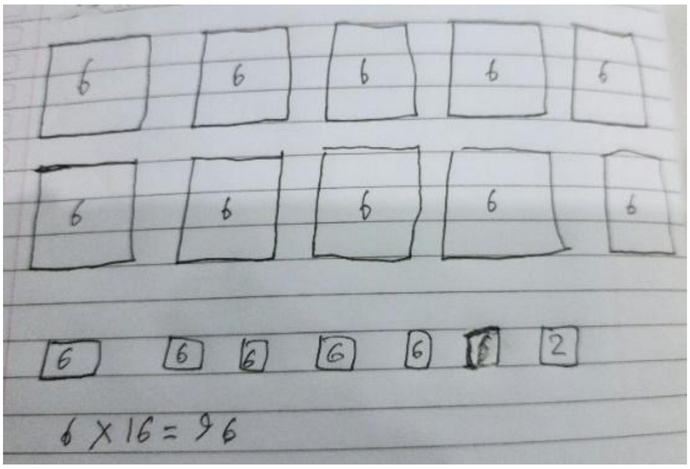

Figure 8. Repeated addition as multiplication

Overall, the students' solutions reveal a range of reasoning strategies that reflect their varying levels of mathematical development. Students with less formal mathematical knowledge tended to rely on concrete, visual strategies (e.g., drawing individual fruits), while those with more advanced reasoning skills used symbolic representations, repeated addition, and multiplicative reasoning. This progression suggests that as students' mathematical understanding deepens, their solution methods become more efficient and abstract, indicating a higher level of mathematical reasoning.

Analysis of Students' Works on Rice Problem

Mathematical reasoning involves the ability to identify relationships among mathematical ideas and to apply these relationships to solve novel problems (Alexander et al., 1997; García-García & Dolores-Flores, 2021; Hwang & Ham, 2021). Even at its most basic level, mathematical reasoning requires more than simply following procedural instructions. It involves the establishment of correspondences between physical or symbolic representations and the abstract concepts they express—a process that relies heavily on analogical reasoning (Sari et al., 2024; Aljura et al., 2025).

Zulfani's Solution: Visual Representation and Mental Fraction Tracking

The solutions produced by students in this study exemplify these theoretical perspectives. One notable case is Zulfani's work presented in Figure 9, which offers a clear example of high-level mathematical reasoning. Although his solution contained few formal mathematical symbols, it demonstrated a deep conceptual understanding of division by fractions and a creative application of visual reasoning.

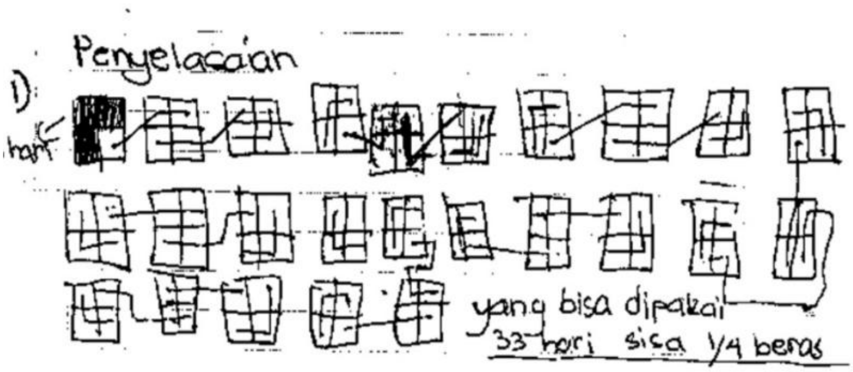

Figure 9. Zulfani's solution to the rice problem

Zulfani represented the 25 kilograms of rice as 25 squares, each square symbolizing 1 kilogram. His decision to use squares was mathematically advantageous, as squares are easily partitioned into four equal parts, enabling each part to represent $\frac{1}{4}$ kilogram. In his solution, Zulfani shaded three out of four parts to represent the $\frac{3}{4}$ kilograms consumed daily, and he annotated each shaded portion with the label "1 day."

Rather than performing formal fraction addition symbolically, Zulfani tracked consumption visually by moving from square to square, mentally combining fractional parts (e.g., $\frac{1}{4}+\frac{2}{4}=\frac{3}{4}$) to represent a day's consumption. Through this systematic approach, he concluded that the rice supply would last 33 days, leaving $\frac{1}{4}$ kilogram unused. This approach bypassed the need for algorithmic fraction addition and instead relied on intuitive reasoning supported by a visual model.

This solution reflects Junaid's (2022) observation that visual proofs can convey mathematical truths without sacrificing rigor, and it illustrates how simple representations can express sophisticated reasoning. In this sense, Zulfani's solution embodies Leonardo da Vinci's maxim that "*simplicity is the ultimate sophistication*."

Kalnako's Solution: Implicit Use of Fraction Addition

Kalnako's work presented in Figure 10 demonstrated a different but equally insightful approach. She began by reasoning that 25 kilograms of rice would last for 25 days if one kilogram were consumed daily, leaving $\frac{1}{4}$ kilogram remaining per day. She then determined how many additional days could be obtained from the daily leftovers. Without explicitly performing formal fraction addition, she reasoned that three leftover quarters make one whole kilogram ($\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4}$), which corresponds to one additional day of cooking.

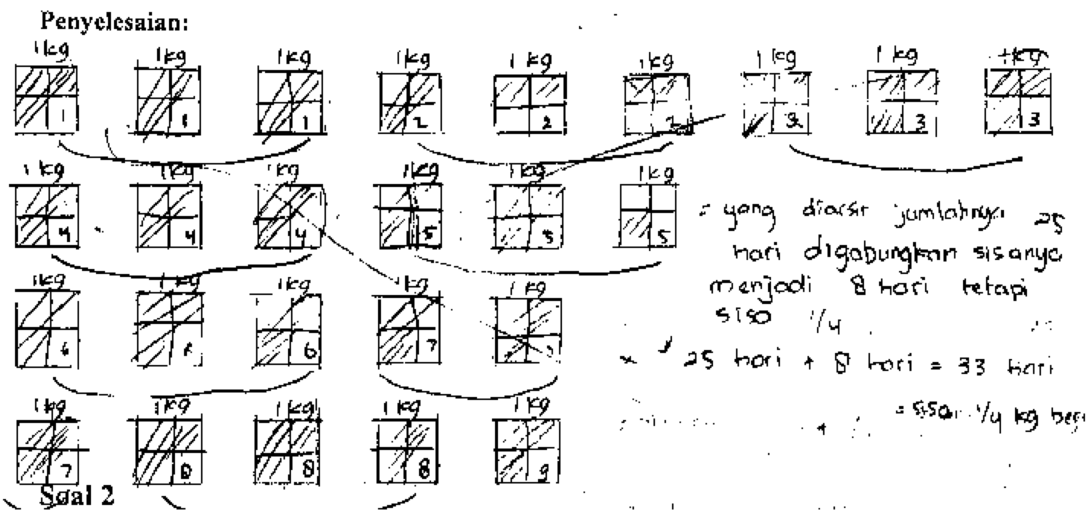

Figure 10. Kalnako's solution to the rice problem

Her solution involved systematically labeling each day on her diagram: the first three squares were marked with "1," representing the first full day from remainders, the next three with "2," and so forth. On the 25th square, she recorded "9," showing that after the ninth group, a $\frac{1}{4}$-kilogram portion still remained. She then combined the results, concluding that the rice would last 33 full days with $\frac{1}{4}$ kilogram left over. Kalnako's reasoning demonstrates an implicit understanding of fractional composition and provides further evidence that students can successfully generalize and extend their reasoning without relying on formal procedures.

Dina's Solution: Formal Fraction Operations

In contrast to the primarily visual approaches above, Dina's work as shown in Figure 11 revealed a strong command of formal fraction operations. Her solution included symbolic calculations, such as $\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}=3$, which she then used as a basis for completing the problem. Dina's solution illustrates how procedural fluency, when combined with conceptual understanding, allows students to connect formal operations with meaningful problem contexts.

Problem 1
Father bought a 25-kg bag of rice. Each day, Mother cooks $\frac{3}{4}$ kg of rice. For how many days will the rice last?

So, the rice that Father purchased will be sufficient for 33 days

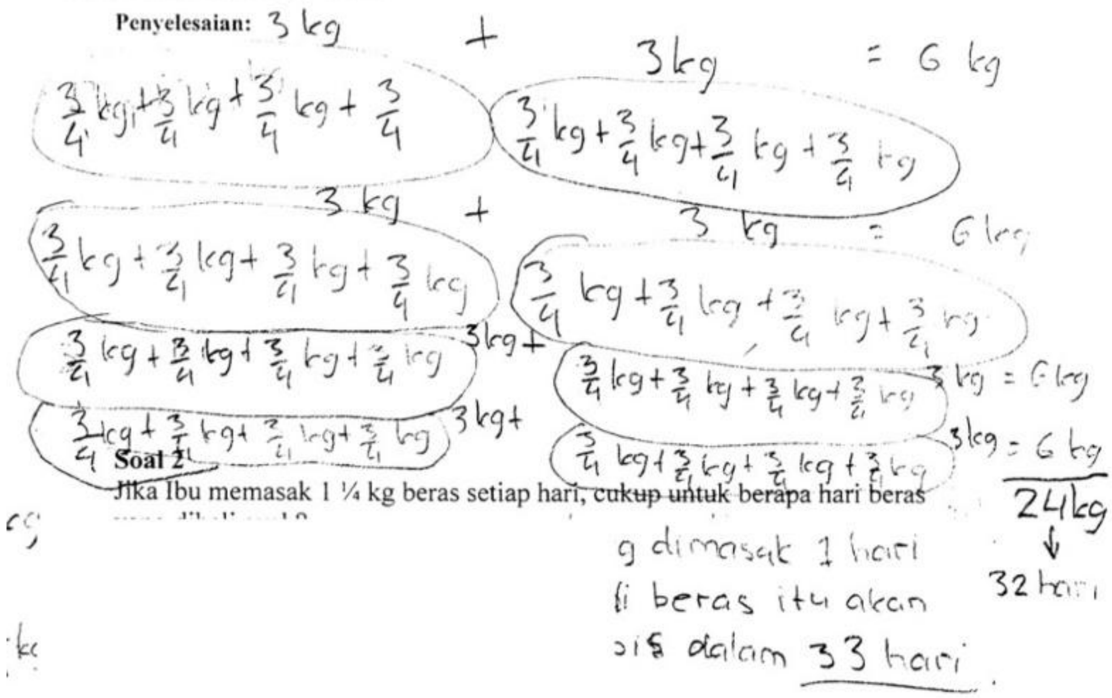

Figure 11. Dina's solution to the rice problem

This finding supports Russell's (1999) characterization of mathematical reasoning as the development, justification, and application of mathematical generalizations that form a network of interconnected knowledge. Dina's solution shows how a student can flexibly move between conceptual understanding, symbolic representation, and real-world application to reach a correct and justifiable solution.

The three examples above demonstrate that students employ diverse reasoning strategies—ranging from visual modeling to symbolic manipulation—depending on their prior knowledge and mathematical experiences. Zulfani and Kalnako relied primarily on informal, visual approaches that enabled them to solve a complex fraction division problem without formal computation. Dina, on the other hand, applied formal fraction addition to reach the solution efficiently. Together, these findings reinforce the view that mathematical reasoning is not merely the mechanical execution of arithmetic operations but rather the dynamic use of conceptual knowledge, representations, and strategies to make sense of and solve problems.

Interpretation of Students' Reasoning in the Presented Problem

Analysis of students' solutions to the dragon fruit problem reveals clear evidence of emergent deductive thinking. Many students employed an implicit if–then logical structure: if each box holds six dragon fruits, then sixteen boxes are required to hold ninety-six fruits, and the remaining two fruits must be placed in a seventeenth box. This reasoning demonstrates their ability to generalize from a contextual situation and to organize their solution steps coherently.

Notably, several students were able to carry out repeated addition symbolically, representing dragon fruits numerically rather than visually. A small number of students went further, recognizing repeated addition as a form of multiplication, thereby using multiplication as a tool for confirmation (e.g., $6 \times 16 = 96$). These findings indicate that students possess what might be described as latent mathematical competencies—powers of imagination, symbolization, conjecture, and generalization—that they express when given an open opportunity to reason through problems. For instance, representing one kilogram with a square or counting by sixes using a self-constructed number line are examples of students inventing their own notations to support problem solving.

The findings align with a consensus in the mathematics education literature that mathematical reasoning forms the foundation for understanding and applying mathematical concepts. Scholars have emphasized the need to strengthen students' reasoning abilities by engaging them in activities such as investigation, representation, conjecture, and justification (Clements et al., 2003; NCTM, 2000; Widjaja et al., 2021; Russell, 1999; Mukuka et al., 2023). According to the Principles and Standards for School Mathematics (NCTM, 2000), mathematics learning is most effective when it continually promotes higher-order thinking and reasoning. Reasoning may thus be seen as the "soil" in which mathematical understanding takes root and grows. When students learn to reason mathematically, they develop the ability to transfer knowledge to new situations, construct connections among concepts, and acquire a foundation for future learning.

Teachers' reflections on the student work were highly positive. Several noted that students' mathematical reasoning often differs from that of teachers, providing unique insights into how children make sense of problems. They also recognized the importance of allowing students to explore multiple solution paths and acknowledged that teachers should not be the sole source of mathematical knowledge in the classroom. Strengthening teachers' understanding of students' reasoning processes can be a powerful lever for professional growth (Hacıeminoğlu et al., 2022; Raza, 2024). As suggested by Fonseca (2018), creating learning environments that support students in organizing their work, conjecturing, and justifying their conclusions fosters more consistent reasoning and valid mathematical proofs. These experiences deepen students' conceptual understanding and prepare them for more advanced mathematical thinking.

CONCLUSION

This study provides evidence that students' mathematical reasoning abilities are diverse and shaped by multiple factors, including prior learning experiences and initial mathematical proficiency. The analysis of students' work on whole-number and fractional division problems revealed a continuum of reasoning strategies, ranging from concrete, visual approaches (e.g., drawing boxes and counting individual items) to more abstract representations (e.g., repeated addition, number-line jumps, and multiplication as a confirmation strategy). These findings suggest that when students are given opportunities to engage with contextual problems and represent their thinking using self-constructed notations, they display emergent deductive reasoning, make conjectures, and construct meaningful mathematical generalizations. In line with previous research (Baroody & Coslick, 1998; NCTM, 2000; Russell, 1999), this study underscores that mathematical reasoning serves as a foundation for conceptual understanding and should be nurtured consistently throughout students' mathematical development.

Furthermore, this study also was conducted in a single primary school with two classes and a limited number of participants, which restricts the generalizability of the findings. Moreover, the qualitative design, while valuable for capturing rich descriptions of students' reasoning, does not allow for strong causal inferences regarding the effects of instructional interventions on mathematical reasoning development. Additional variables—such as teacher practices, students' prior achievement levels, and socio-cultural factors—may have influenced the outcomes but were not systematically controlled in this study. Future research should employ mixed-methods designs or longitudinal approaches to more robustly examine the interaction between instructional approaches, students' prior knowledge, and the development of mathematical reasoning over time.

Finally, the findings of this research have important implications for mathematics teaching and teacher education. Designing lessons around contextual problems appears to stimulate students' reasoning processes and enables them to construct knowledge actively, which may enhance their long-term mathematical proficiency. We recommend that future studies systematically investigate the impact of various instructional models—such as problem-based learning, realistic mathematics education, and inquiry-oriented approaches—on students' reasoning skills across diverse educational settings. Furthermore, professional development programs for teachers should emphasize strategies for eliciting, interpreting, and building upon students' reasoning, as this can support teachers in creating classrooms where reasoning and proof are central to instruction. Strengthening students' reasoning from the earliest grades not only promotes deeper mathematical understanding but also prepares them to approach complex and abstract mathematical ideas with confidence.

Acknowledgments

We would like to express our profound gratitude to Tim PMRI FKIP ULM for his invaluable support during the preparatory phase and in the classroom experiment. It is evident that the successful execution of the experiment was made possible by the collaborative efforts of the team, the principal, and the teachers at Elementary School of Antasan Besar 7, in Banjarmasin, Indonesia. We would like to express our gratitude for the continued support provided throughout the course of the project.

Declarations

Author Contribution	:	SH: Creating Problems, Classroom Experiment, Data Analysis, and Composing Article. MD: Creating Problems, Classroom Experiment, Data Analysis, Comment, and Revision to the Composition. K: Classroom Experiment and Discussion with Teachers. TH: Observation and Data Collection.
Funding Statement	:	We would like to express our gratitude to P4MRI-ULM and Lambung Mangkurat University for their financial support, which was instrumental in facilitating the visit of Dr. Maarten Dolk from the Freudenthal Institute in the Netherlands to Indonesia. This visit took place during the research.
Conflict of Interest	:	The authors declare no conflict of interest.
Additional Information	:	Additional information is available for this paper.

REFERENCES

Alexander, P. A., White, C. S., & Daugherty, M. (1997). Children's use of analogical reasoning in early mathematics learning. In L. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 117 – 147). Mahwah, NJ: Lawrence Erlbaum Associates.

Aljura, A. N., Retnawati, H., Dewanti, S. R., Kassymova, G. K., Sotlikova, R., & Septiana, A. R. (2025). Mathematical reasoning and communication word problems with mathematical problem-solving orientation: A relation between the skills. *Journal on Mathematics Education, 16*(2), 529–558. https://doi.org/10.22342/jme.v16i2.pp529-558

Baroody, A. J., & Coslick, R. T. (1998). *Fostering children's mathematical power: An investigative approach to K-8 mathematics instruction*. Routledge. https://doi.org/10.4324/9781410602084

Bragg, L. A., Loong, E. Y. K., Widjaja, W., Vale, C., & Herbert, S. (2015). Promoting reasoning through the magic V task. *Australian Primary Mathematics Classroom, 20*(2), 10-14. https://search.informit.org/doi/abs/10.3316/INFORMIT.330329444881101

Callingham, R., & Siemon, D. (2021). Connecting multiplicative thinking and mathematical reasoning in the middle years. *The Journal of Mathematical Behavior, 61*, 100837. https://doi.org/10.1016/j.jmathb.2020.100837

Carayannis, E. G., Christodoulou, K., Christodoulou, P., Chatzichristofis, S. A., & Zinonos, Z. (2022). Known unknowns in an era of technological and viral disruptions—Implications for theory, policy, and practice. *Journal of the knowledge economy, 13*(1), 587-610. https://doi.org/10.1007/s13132-020-00719-0

Chiu, S., & Tron, M. O. (2004). Classroom discourse and the development of mathematical and analogical reasoning. In L. English (Ed.), *Mathematical and analogical reasoning of young learners*. Lawrence Erlbaum Associates.

Clements, D. H., Sarama, J., & DiBiase, A. M. (2003). *Engaging young children in mathematics: Findings of the 2000 National Conference on Standards for Preschool and Kindergarten Mathematics Education*. Lawrence Erlbaum Associates.

Ernie, K., LeDocq, R., Serros, S., & Tong, S. (2023). Mathematical reasoning: Challenging students' beliefs about mathematics. In R. A. R. Gurung, N. L. Chick, and A. Haynie (eds.), *Exploring Signature Pedagogies* (pp. 260-279). Routledge. https://doi.org/10.4324/9781003444732

Fonseca, L. (2018). Mathematical reasoning and proof schemes in the early years. *Journal of the European Teacher Education Network, 13*, 34-44. https://drive.google.com/file/d/1zc18YZ7V1mNSzEbrEMN036fCSd5lyJys/view

García-García, J., & Dolores-Flores, C. (2021). Exploring pre-university students' mathematical connections when solving Calculus application problems. *International Journal of Mathematical Education in Science and Technology, 52*(6), 912-936.

Ginsburg, H. P., Pappas, S., & Seo, K.-H. (2001). Everyday mathematical knowledge: Asking young children what is developmentally appropriate. In S. L. Golbeck (Ed.), *Psychological perspectives on early childhood education: Reframing dilemmas in research and practice* (pp. 181–219). Lawrence Erlbaum Associates Publishers.

Hacıeminoğlu, E., Yıldız, N. G., & Şeker, R. (2022). Factors related to cognitive reasoning of pre-service teachers' science process skills: Role of experiments at home on meaningful learning. *Sustainability, 14*(13), 7703. https://doi.org/10.3390/su14137703

Herbert, S., & Williams, G. (2023). Eliciting mathematical reasoning during early primary problem solving. *Mathematics Education Research Journal, 35*(1), 77-103. https://doi.org/10.1007/s13394-021-00376-9

Hsu, F. H. (2022). *Behind Deep Blue: Building the computer that defeated the world chess champion*. Princeton University Press.

Hwang, J., & Ham, Y. (2021). Relationship between mathematical literacy and opportunity to learn with different types of mathematical tasks. *Journal on Mathematics Education, 12*(2), 199-222. http://doi.org/10.22342/jme.12.2.13625.199-222

Junaid, M. (2022). *Mathematical intelligence: A story of human superiority over machines*. Simon and Schuster.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. National Academy Press.

Lee, S., Lee, M., & Lee, S. (2023). What if artificial intelligence become completely ambient in our daily lives? Exploring future human-AI interaction through high fidelity illustrations. *International Journal of Human–Computer Interaction, 39*(7), 1371-1389. https://doi.org/10.1080/10447318.2022.2080155

Mukuka, A., Balimuttajjo, S., & Mutarutinya, V. (2023). Teacher efforts towards the development of students' mathematical reasoning skills. *Heliyon, 9*(4), e14789. http://dx.doi.org/10.1016/j.heliyon.2023.e14789

NCTM (2000). *Principles and standards for school mathematics*. NCTM.

Panaoura, R., & Nitsiou, C. (2023). Children's informal learning in mathematics through parental involvement with play-based activities: A nonformal training program. *Education Sciences, 13*(9), 870. https://doi.org/10.3390/educsci13090870

Papandreou, M., & Tsiouli, M. (2022). Noticing and understanding children's everyday mathematics during play in early childhood classrooms. International Journal of Early Years Education, 30(4), 730-747. https://doi.org/10.1080/09669760.2020.1742673

Prabhu, M., & Premraj, J. A. (2025). Artificial consciousness in AI: A posthuman fallacy. *AI & Society, 40*(4), 2995-3008. https://doi.org/10.1007/s00146-024-02061-4

Pramudiani, P. (2023). Professional development for supporting primary school teachers in promoting students' mathematical reasoning using realistic mathematics education. *Dissertation*. Indonesia University of Education.

Ramlan, A. M., Budayasa, I. K., & Rahaju, E. B. (2025). Proportional reasoning in the artisan personality type: A case study of high school students in trigonometry ratios. *Journal on Mathematics Education, 16*(1), 73–90. https://doi.org/10.22342/jme.v16i1.pp73-90

Raza, M. (2024). AI in Education: Balancing Benefits and Challenges Through Professional Development of Teachers. *Qlantic Journal of Social Sciences, 5*(4), 168-174. https://doi.org/10.55737/qjss.v-iv.24071

Resnick, L. B. (1989). Developing mathematical knowledge. *American Psychologist, 44*(2), 162–169. https://doi.org/10.1037/0003-066X.44.2.162

Rianasari, V. F., & Guzon, A. F. H. (2024). Designing learning trajectory to support preservice mathematics teachers' skills in creating and implementing realistic mathematics tasks. *Journal on Mathematics Education, 15*(3), 701–716. https://doi.org/10.22342/jme.v15i3.pp701-716

Russell, S. J. (1999). Mathematical reasoning in the early grades. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning, K–12* (pp. 22 – 36). National Council of Teachers of Mathematics.

Sari, R. N., Rosjanuardi, R., Isharyadi, R., & Nurhayati, A. (2024). Level of students' proportional reasoning in solving mathematical problems. *Journal on Mathematics Education, 15*(4), 1095–1114. https://doi.org/10.22342/jme.v15i4.pp1095-1114

Siswantari, Sabon, S. S., Listiawati, N., Wirda, Y., Zulkardi, & Riyanto, B. (2025). Bridging mathematics and communication: Implementing realistic mathematics education principles for skill development. *Journal on Mathematics Education, 16*(2), 729–752. https://doi.org/10.22342/jme.v16i2.pp729-752

Sousa, D. A. (2015). *How the brain learns mathematics* (Second Edition). A Sage Publishing Company.

Sun, J., Zheng, C., Xie, E., Liu, Z., Chu, R., Qiu, J., ... & Li, Z. (2025). A survey of reasoning with foundation models: Concepts, methodologies, and outlook. *ACM Computing Surveys, 57*(11), 1-43. https://doi.org/10.1145/3729218

Torres-Peña, R. C., Peña-González, D., Lara-Orozco, J. L., Ariza, E. A., & Vergara, D. (2025). Enhancing numerical thinking through problem solving: A teaching experience for third-grade mathematics. *Education Sciences, 15*(6), 667. https://doi.org/10.3390/educsci15060667

Widjaja, W., Vale, C., Herbert, S., Loong, E. Y., & Bragg, L. A. (2021). Linking comparing and contrasting, generalising and justifying: A case study of primary students' levels of justifying. *Mathematics Education Research Journal, 33*(2), 321-343. https://doi.org/10.1007/s13394-019-00306-w

Xu, J., & Cai, D. (2024). The cognitive foundations of different hierarchical levels of mathematical skills in primary school children: extending the mathematics pathways model. *European Journal of Psychology of Education, 39*(4), 3919-3941. https://doi.org/10.1007/s10212-024-00823-8