Journal on Mathematics Education
Volume 15, No. 3, 2024, pp. 859-882

Designed activities on developing mental strategies using alms value-based learning trajectory in elementary school

Neni Mariana*, Febriarsita Eka Sasmita

Department of Elementary Teacher Education, Universitas Negeri Surabaya, Surabaya, Indonesia
*Correspondence: nenimariana@unesa.ac.id

Received: 19 February 2024 | Revised: 16 May 2024 | Accepted: 19 June 2024 | Published Online: 10 August 2024

Abstract

Mental strategies are recognized as an effective approach for solving arithmetic operations such as addition and subtraction. This study focuses on developing and implementing instructional activities aimed at enhancing students' mental strategy skills through the Indonesian Realistic Mathematics Education (IRME) approach. The research employed a design-based methodology involving 2nd-grade students from an Islamic-integrated school in Surabaya. The study followed a research design that emphasizes the development of local instructional theory, encompassing three key phases: the preparatory phase (initial design development), the experimental phase (classroom teaching experiments), and the retrospective analysis phase. Data collection methods included documentation, field notes, and interviews. In the first trial, three instructional activities were conducted. Based on the analysis of the first trial, the number of activities in the second trial was increased to four: "Let's Get Used to Alms Every Day," "Don't Forget Routine Alms This Week," "Set Your Own Pocket Money," and "Let's Give Away Your Alms to Break Mr. Samad's Fast." These activities were designed to achieve specific mathematical objectives, enabling students to understand and solve addition and subtraction problems using strategies such as skip counting, borrowing, and saving. The retrospective analysis concluded that students successfully learned to perform addition and subtraction operations on both monetary values and ordinary numbers within the designed learning trajectory.

Keywords: Addition and Subtraction, Design Research, Indonesian Realistic Mathematics Education, Mental Strategies

How to Cite: Mariana, N., & Sasmita, F. E. (2024). Designed activities on developing mental strategies using alms value-based learning trajectory in elementary school. *Journal on Mathematics Education, 15*(3), 859-882. http://doi.org/10.22342/jme.v15i3.pp859-882

Arithmetic operations are foundational mathematical concepts that elementary school students must master (Principles, 2000). In elementary mathematics, four primary arithmetic operations—addition, subtraction, multiplication, and division—are essential for students to learn (Cooke, 2001). Before progressing to the concepts of multiplication and division, students must first develop a solid understanding of addition and subtraction. These two operations serve as the building blocks for other arithmetic processes, as multiplication is fundamentally based on repeated addition, and division is closely related to subtraction (Freudenthal, 2012; Principles, 2000).

According to curriculum documents in Indonesia, students are expected to master addition and subtraction by the second grade of elementary school. However, many students in Indonesia still struggle with addition, particularly when working with large numbers. Nengsih and Pujiastuti (2021) have noted that this issue persists across both lower and upper grades. The difficulty arises because students often rely on counting, using their fingers, or recalling from memory (Van Nes & De Lange, 2007). When dealing with numbers over 100, students find it challenging to rely solely on their memory. Some researchers suggest that the procedural and conceptual knowledge students have previously acquired plays a more significant role in their ability to perform these operations (Beishuizen, 1993; Beishuizen et al., 1997; Resnick, 1986).

Students can develop procedural and conceptual knowledge by enhancing their mental arithmetic skills and employing mental strategies (Ashcraft, 1985; Treffers, 1991). Conceptual knowledge refers to the understanding that connects various pieces of information, enabling students to grasp the relationships and interconnections between different concepts (Carpenter, 2013; Crooks & Alibali, 2014). In contrast, procedural knowledge involves the steps or methods a student uses to achieve learning objectives, including the rules, skills, and actions they apply to solve problems (Carpenter, 2013; Crooks & Alibali, 2014). Procedural knowledge can be considered a component of prior knowledge. For example, in mathematics education, when students first learn to count, they often use their hands and fingers as a procedural tool (Hiebert & Lefevre, 2013).

Mental arithmetic, or mental calculation, involves performing mathematical computations in the mind without the assistance of any tools, such as calculators, writing instruments, or even fingers. This skill is invaluable, as it allows students to conduct mathematical calculations using mental techniques such as visualization, memorization, and pattern recognition (Proulx, 2013). One of the key advantages of developing mental arithmetic is its contribution to enhancing students' overall arithmetic abilities. Regular practice of mental math can improve recall, focus, and problem-solving skills. Furthermore, it proves beneficial in situations where quick calculations are required, such as during tests, competitions, or daily activities (Proulx, 2019).

Mental arithmetic is closely linked to mental strategies, with the former indirectly influencing the latter. Students who excel in mental arithmetic often develop effective mental computation strategies, enabling them to handle a broader range of calculations, including most two-digit computations and simple fractions, decimals, and percentages. For more complex problems, they may still resort to paper, pencil, or calculators (Callingham, 2005). Mental computation involves the ability to use various strategies to solve arithmetic problems with large numbers (Heirdsfield, 2011). Mental strategies refer to the processes students use to work out and arrive at accurate answers or estimates mentally, choosing the most appropriate strategy depending on the numbers involved (Varol & Farran, 2007).

In Indonesia, many students are still unfamiliar with the use of mental strategies in mathematics. This is largely due to the infrequent integration of mental strategies in number learning. Teachers often rely on manual and conventional methods, emphasizing standard algorithms and procedures without adequately teaching the underlying concepts (Armanto, 2002). Given this background, there is a pressing need to develop a learning design or trajectory that can stimulate students' mental strategies, particularly in understanding and performing addition and subtraction with money.

A learning trajectory is essential because students naturally follow a progression as they learn and develop (Sztajn et al., 2012). Just as children progress from crawling to walking, running, and jumping with increasing skill, they also follow a natural progression in their learning processes, particularly in mathematics. By understanding these natural developmental patterns, teachers can create an effective and appropriate learning environment. These patterns form the basis of learning trajectories, which are particularly relevant to mental computation activities that move from simple to complex processes.

Learning trajectories guide teachers in designing instructional plans. They provide a framework for determining and formulating learning objectives, enabling teachers to make informed decisions about the strategic steps needed to achieve these goals (Clements & Sarama, 2012). Furthermore, learning trajectories help students develop deeper conceptual understanding by building on their prior knowledge (Gravemeijer, 2004; Gravemeijer & Terwel, 2000). In this context, the application of learning trajectories often involves the Realistic Mathematics Education (RME) approach, which connects students' experiences to their existing knowledge (Yackel & Cobb, 1996).

The Indonesian adaptation of Realistic Mathematics Education (RME) is known as Pendidikan Matematika Realistik Indonesia (PMRI). PMRI emphasizes the integration of Indonesian cultural values into the RME framework, which utilizes real-world contexts relevant to students' everyday lives (Siswono, 2006; Soedjadi, 2014). The core principle of RME is that mathematics instruction should begin with contextual, real-life situations, making mathematical concepts more accessible and understandable for students. By starting with familiar, concrete contexts, students are better equipped to grasp abstract mathematical ideas, reducing the difficulty of learning complex concepts.

In the PMRI approach, the learning process is paramount, emphasizing that students must engage actively in the learning process, discovering mathematical concepts and relationships through exploration and interaction. Teachers play a crucial role in guiding students, using various cultural contexts that resonate with students' backgrounds to facilitate understanding. This method not only makes mathematics more relatable but also helps students develop a deeper and more meaningful understanding of mathematical concepts.

In this study, researchers utilized the context of alms (*infaq*) as a pedagogical tool, as it is deeply embedded in the daily lives of the students at the participating school. *Infaq*, derived from the Arabic word "*anfaqa*," meaning to give wealth or a portion of one's sustenance voluntarily, is an act of worship in Islam that carries both divine significance and social value (*Maliyah Ijtimah'iyyah*). In the Islamic context, *infaq* refers to the act of giving wealth or assistance to others as an expression of social care, fostering community solidarity and providing support to those in need, such as the poor, orphans, or people in distress.

The context of *infaq* was chosen because it resonates with the participants—second-grade students at an integrated Islamic elementary school—where the practice of *infaq* is encouraged to instill the value of sharing. This context is also directly relevant to the mathematical concepts of addition and subtraction, as *infaq* involves the act of reducing one's assets to give to others, aligning with the arithmetic operations of subtracting and adding wealth (Armanto, 2002). Additionally, in the Islamic tradition, it is implicitly taught that those who give in *infaq* will receive sustenance from Allah SWT.

The use of a religious context, such as *infaq*, is relatively novel in elementary mathematics education in Indonesia, where cultural contexts are more commonly employed. Despite its familiarity and significance in students' lives, religious contexts have not been widely explored in mathematical instruction. By integrating the context of *infaq*, this research aims to explore the computational strategies that emerge when students engage with the concepts of addition and subtraction through this Islamic practice. The study seeks to understand how this context can enhance students' understanding of these arithmetic operations.

METHODS

Research Procedure

To address the objectives outlined and evaluate the instructional design that supports student problem-solving, we propose employing the Gravemeijer Design Research Model (Gravemeijer & Cobb, 2006).

Design research focuses on developing local teaching theories and is characterized by five key traits: intervention character, process-orientedness, reflective components, cyclical nature, and theory orientation (Gravemeijer & van Eerde, 2009; Prahmana & Kusumah, 2017; Prediger et al., 2015). This approach was chosen for its systematic and flexible strategy for improving classroom instruction through collaboration between researchers and educators to develop effective learning designs (Gravemeijer, 2004).

The design research process comprises three stages, namely the preparation phase, conducting teaching experiments, and conducting a retrospective analysis (Gravemeijer, 2004; Gravemeijer & Cobb, 2006). Each phase is designed to ensure a comprehensive and iterative approach to refining instructional practices, as detailed below:

1. Preliminary Research

   The preliminary design stage involves developing learning activities and assessment tools to evaluate the learning process. This research began with a literature review on the concepts of addition and subtraction, the Indonesian Realistic Mathematics Education (IRME) approach, and the analysis of these operations within the Indonesian mathematics curriculum. The goal was to formulate conjectures about students' thinking processes. During this phase, a Hypothetical Learning Trajectory (HLT) was created as both a design framework and an assessment instrument. The HLT outlines the objectives of the activities, the activities themselves, and anticipated student responses. In this study, the HLT includes four activities aimed at deepening students' understanding of addition and subtraction. The HLT also serves to guide the direction and focus of the research analysis (Risdiyanti & Prahmana, 2020). Throughout this phase, the HLT is refined through thought experiments, where hypotheses about student thinking are tested from the perspective of experienced teachers. Adjustments to the HLT are made based on findings from each stage of the research (Prahmana, 2017).

2. Design Experiment

   The design experiment phase involves anticipating and verifying the learning design, specifically the Hypothetical Learning Trajectory (HLT) developed during the preliminary research phase (Mariana, 2009). In this phase, the learning trajectory created in the earlier stages is tested to identify the mental strategies students employ when learning subtraction and addition. The design experiment consists of two cycles. This study was conducted at SD-IT At-Taqwa in Surabaya, focusing on classes II-E and II-F. The principal recommended these classes due to the teachers' ability to effectively manage and condition the students. The first cycle was carried out with 9 students from class II-E, and the second cycle involved 16 students from class II-F. The students were selected by their teacher based on their active participation and proficiency in daily classroom activities. Each cycle lasted between 2 and 2.5 hours. The insights gained from these instructional activities will inform the design or modification of future learning activities and contribute to new conjectures about students' mental strategies (Gravemeijer, 2004).

3. Retrospective Analysis

   The retrospective analysis phase aims to ensure that the local instructional theory is well-developed and of high quality (Van den Akker et al., 2006). During this phase, data obtained from the design experiment are evaluated by comparing the initial conjectures and the Hypothetical Learning Trajectory (HLT) with the actual outcomes observed during the implementation of the learning trajectory (Gravemeijer & Cobb, 2006). This involves assessing student difficulties and strategies identified during the study and analyzing their responses to the designed activities.

Based on this analysis, the researcher refines the HLT and revises the learning activities. The second cycle of the design experiment is then conducted, incorporating improvements based on the findings from the first cycle. The researcher re-analyzes the classroom data to further refine the HLT and adjust the activities as needed. The focus is on achieving the objectives of each activity and using the HLT as a guide for the analysis.

Data Gathering Techniques

In design research, appropriate data collection techniques are essential to achieve the study's objectives. The implementation of these techniques typically involves:

1. Field Notes

   Field notes are a valuable tool for qualitative researchers seeking to gather data through observations and interviews conducted in the field. Researchers record their observations in the field and then compile these notes upon returning to their office or residence. In this study, the field notes are descriptive, providing a detailed account of the observation context, participants, and actions relevant to the research focus. Descriptive field notes include several components: physical descriptions of the setting, reconstructions of dialogues, descriptions of the physical environment, notes on significant events, descriptions of activities, and observations of behavioral interactions (Moelong, 2012). For this study, field notes were used to document student strategies for problem-solving, difficulties encountered by students during the learning process, and other relevant occurrences observed during the instructional sessions.
2. Interviews

   In this study, unstructured interviews are employed, meaning that researchers do not follow a predetermined set of guidelines. These interviews are conducted as needed to gather additional information not captured through field notes (Hopf, 2004). Researchers conduct interviews with teachers to discuss anticipated learning outcomes and with students to gain insights into their experiences and perceptions of the learning process.

The Quality Standards

To ensure the validity of the data in this research, which does not involve numerical data, we employ techniques focused on maintaining the quality and credibility of the study. These techniques, often referred to as quality standards, are essential for validating the research findings (Mariana, 2019). In design research, internal validity is primarily determined by the quality of the data collection methods and the interpretation processes that lead to conclusions. Therefore, the quality of the Hypothetical Learning Trajectory (HLT) and field documentation can significantly influence the validity of the researchers' conclusions (Van den Heuvel-Panhuizen & Drijvers, 2020).

In design research, two key methods are used to assess the validity of the research: Data Triangulation and Cross-Interpretation. Data Triangulation involves examining relationships and correlations between data and findings collected through various techniques. It helps ensure the robustness of the results by comparing different data sources (Prahmana, 2017). Cross-Interpretation involves consulting with experts to minimize subjectivity and enhance the credibility of the findings obtained from data collection.

Triangulation, in particular, employs multiple perspectives to validate data, including those of teachers, participant observers, and external experts. Internal validity is further ensured by engaging in discussions with the teacher and participant observers after each learning experiment. The outcomes of these discussions are documented in analysis notes and reviewed collaboratively with the teacher and observers. Any discrepancies or additional insights are addressed through further discussions to refine the analysis and ensure accuracy.

Simon and Tzur (2012) identify two key reliability techniques in design research, namely internal reliability and external reliability. Internal reliability is achieved by organizing and categorizing data according to its type, including video recordings and photographs. This involves discussions with research colleagues who serve as observers during the design trial. Internal reliability ensures that the research design is robust, with tasks and roles clearly defined and cooperation effectively established with teachers and schools. It also involves consistency in subjective interpretations of data, including conclusions drawn from retrospective analysis. Feedback from teachers, participant observers, and experts contributes to a thorough analysis of the learning process observed in the classroom. Furthermore, external Reliability is fulfilled by documenting all research outcomes, both successes and failures, and providing a detailed explanation of the procedures followed, the theoretical frameworks used, and the rationale behind each decision. This includes reporting on the rationale for context selection in the development of local instructional theory. Validity checks are performed to ensure that the collected data aligns with the research focus and objectives.

RESULTS AND DISCUSSION

The results of this study include the development of a learning trajectory designed for testing and the findings from the teaching experiments conducted over both cycles. The learning trajectory, or Hypothetical Learning Trajectory (HLT), was developed using the context of alms, specifically tailored for second-grade elementary students. The first context focuses on solving problems related to addition, subtraction, whole numbers, time, length, weight, and money in daily activities at home, school, or playgrounds. It involves examining and verifying solutions and expressing them mathematically. The second context similarly addresses problems in these areas but emphasizes verification and articulation of mathematical sentences in students' own words.

According to the basic competency standards, the mathematical goals of the HLT are to enable students to calculate the total amount of money using the "addition" mental strategy and to determine the remaining money using the "borrowing" mental strategy. The class teacher reported that students had grasped the basic concepts of addition and subtraction but noted that this activity effectively reinforced their understanding of these concepts, particularly in relation to stacked addition and subtraction involving borrowing and saving strategies.

Arithmetic operations, including addition, subtraction, multiplication, and division, are often taught using mental strategies. Mental strategies, or mental computation strategies, represent the most common form of computation used in everyday life. These strategies are essential for quick calculations and estimations, going beyond simple mental arithmetic. Mental computation involves the process of determining exact or approximate answers mentally, allowing students to choose from a range of strategies based on the numbers involved. As students expand their repertoire of strategies, they tend to select those that are more efficient and effective for their needs.

In line with the indicators and objectives set by educators, students are expected to solve problems involving addition and subtraction of money, particularly when dealing with amounts exceeding hundreds. To support this, researchers advocate for designing activities that facilitate students' practice of calculation strategies. Arithmetic strategies are crucial for aiding students in learning arithmetic operations. Carpenter and Moser (1984) emphasize that teaching arithmetic operations should focus on developing mental strategies based on students' thought processes. Introducing mental strategies early in elementary education is essential, as it prevents reliance on finger counting (Van Nes & de Lange, 2007). The benefits of mental computation include the development of higher-order thinking, reasoning, and critique skills, as well as a deeper understanding of numbers and operations. Mental computation enhances children's understanding of numerical relationships, helps them make procedural decisions, and fosters the creation of effective calculation strategies.

The use of initial mental strategies can help students develop a broader range of mental computation techniques. The strategies that may emerge from students, based on the researcher's conjecture, include: counting all, counting on from first counting strategy, counting on from larger calculation strategy, using doubles calculation strategy, skip counting calculation strategy, jumping calculation strategy, compensation calculation strategy, bridging through ten calculation strategy, splitting calculation strategy, and pen and paper calculation strategy (Ashcraft & Fierman, 1982; Beishuizen, 1993; Beishuizen et al., 1997; Carpenter et al., 1998; Carpenter & Moser, 1984; Carraher et al., 2006; Geary et al., 2004; Herman, 2001; Karlsson & Johansson, 2018; Macintyre & Forrester, 2003; McIntosh et al., 2005; Reys, 1985, 1986; Reys et al., 1993; Reys et al., 1995; Thompson, 1999; 2010; Varol & Farran, 2007; Widaman et al., 1989).

Furthermore, for subtraction operations, several strategies support practical mental computation through various studies on mental computation and arithmetic operations, demonstrating their effectiveness in helping students perform subtraction tasks more efficiently and accurately, such as counting back calculation strategy, counting back calculation strategy, counting up counting strategy, using doubles counting strategy, skip counting calculation strategy, jumping counting strategy, compensation calculation strategy, bridging through ten calculation strategy, splitting calculation strategy, and pen and paper calculation strategy, whose are supported by various studies in mental computation and arithmetic operations (Ashcraft & Fierman, 1982; Beishuizen, 1993; Beishuizen et al., 1997; Carpenter & Moser, 1984; Carraher et al., 2006; Geary et al., 2004; Herman, 2001; Karlsson & Johansson, 2018; Macintyre & Forrester, 2003; McIntosh et al., 2005; Reys, 1985, 1986; Reys et al., 1993; Reys et al., 1995; Thompson, 1999; 2010; Varol & Farran, 2007; Widaman et al., 1989).

The activities designed in this research aim to encourage students to utilize and become familiar with mental strategies for arithmetic operations. The study resulted in Local Instructional Theories (LITs), which consist of three Hypothetical Learning Trajectories (HLTs). Local Instructional Theory describes and rationalizes the envisioned learning route, outlining a set of instructional activities tailored for a specific topic (Gravemeijer, 2004).

Gravemeijer (2004) highlights two key distinctions between an LIT and an HLT. Firstly, while an HLT focuses on a limited number of instructional activities, an LIT encompasses an entire sequence of learning experiences. Secondly, HLTs are designed with a specific classroom setting in mind, whereas an LIT provides a broader framework that guides the development of HLTs for various instructional contexts. Therefore, the primary differences between LITs and HLTs are their scope and their application within particular classroom environments.

The activities outlined in HLT 1 draw upon the theories of Beishuizen (1993) and Thompson (1999), who emphasize that mental strategies typically emerge from students starting with the simplest forms of calculations that they encounter in their everyday lives. Students are more likely to develop effective mental strategies if these strategies begin with familiar and manageable tasks that do not overwhelm them (Baroody, 1992; Thevenot et al., 2016). HLT 1 incorporates these principles by initiating activities that involve students independently counting money using their own methods. According to Mariana et al. (2018), simple tasks such as calculating the total salary with self-devised strategies, creating shopping lists with estimated prices, calculating total expenditures and remaining funds, allocating remaining resources for basic needs, and converting calculations into fractions, can effectively stimulate students' thinking and help them develop their mental strategies.

In HLT 1, the researcher designed three distinct activities to span across three lessons, each incorporating the alms context to achieve specific mathematical goals. The activities are "Let's Get Used to Alms Every Day," "Don't Forget to Alms Regularly This Week," and "Determine Your Own Pocket Money." Each of these activities is designed to be simple, engaging, and enjoyable, aligning with the overall goals of HLT 1. They aim to promote familiarity with mental strategies through practical and relatable contexts. For a detailed overview, Table 1 summarizes the activities, the theoretical framework supporting each activity, and the corresponding mathematical objectives.

Table 1. Summarizes the activities, underlying theory, and mathematical goal of the first HLT

Underlying Theory	Activities	Mathematical Goal
The mental strategies that most often appear when learning addition in elementary school students are counting all counting strategies, counting on from first counting strategies, counting on from larger strategies, using doubles counting strategies, skip counting strategies, jumping counting strategies, compensation calculation strategies, bridging through ten calculation strategies, splitting calculation strategies, and pen and paper calculation strategies	Let's get used to alms every day	To Perform Addition and Subtraction using skip counting calculation strategy
The calculation strategies that appear in the concept of subtraction calculation operations are counting back calculation strategy, counting back calculation strategy, counting up calculation strategy, using doubles counting strategy, skip counting calculation strategy,	Don't forget to give charity regularly this week	To Perform addition and subtraction using friendly number calculation and splitting strategies

Underlying Theory	Activities	Mathematical Goal
jumping calculation strategy, compensation calculation strategy, bridging through ten calculation strategy, splitting calculation strategy, and pen and paper calculation strategy		
	You can set your allowance by yourself	Perform addition and subtraction using the Borrow and Save calculation strategy

In Activity 1 of HLT 1, students engaged in two main tasks: counting one group's pocket money using the skip counting-on strategy and calculating the remaining pocket money after alms using the skip counting-back strategy. Remarkably, students performed well beyond expectations, demonstrating strong proficiency in these tasks. They were able to solve the problems efficiently and without significant difficulty.

Contrary to the conjecture in HLT 1, which anticipated that students might count items randomly or from smallest to largest, all groups instead counted from the largest amount to the smallest. Notably, the "Great" and "Smart" groups chose to count their members' pocket money individually, as illustrated in the following conversation:

Teacher : Naah according to the fraction of pocket money shown by Ms. Febri on the screen which also corresponds to the amount of pocket money of all of you how do you calculate it.
Azza : Do we have to count as usual?
Teacher : Yes girls, how did you calculate?
Marsya : But my pocket money is 10,000, so my mother usually gives me 5,000 instead of 500 and 2,000.
Teacher : Yes, Marsya but it could be that one day your mom runs out of 5,000 so she gives 500 and 2000.
Azza : We usually count one by one.

Another term that emerged was "skip counting," which students interpreted as a method of quickly adding or subtracting by visually referring to the money media. Students commonly used skip counting by 5s and 10s to sum up their pocket money.

For calculating the remaining 2000, they were able to perform the calculation swiftly due to the relatively small number of alms involved. When counting 500, students also utilized skip counting by 10s, as demonstrated in the following conversation:

Teacher : What if the infaq uses 500 rupiahs?

Reyhana : We can do like this we have 24 coins 500 (while showing the scribble drawing on the paper) directly taken the rest is 2 first then taken 10 so the rest is 12, as shown in Figure 1.

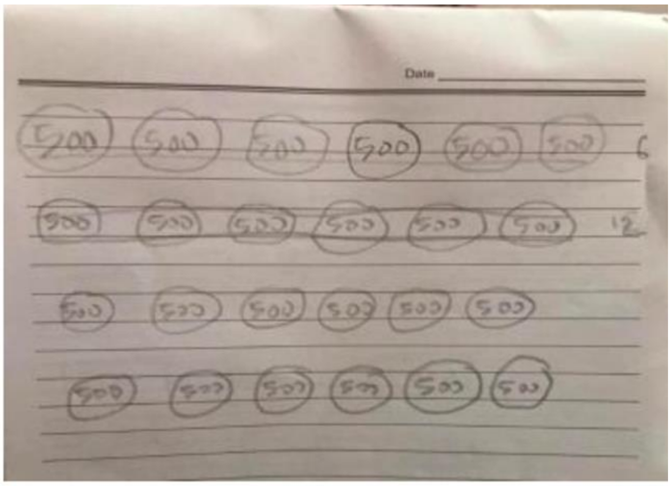

Figure 1. Student's work

Through the first activity, both teachers and researchers aimed to see similar reasoning and understanding in the second activity. The success and strategies observed in the initial task provide a solid foundation for subsequent activities. The evidence gathered will not only serve as a benchmark for the next day's lessons but also inspire other students to apply quick and effective strategies for solving problems related to counting and managing money. This continuity in approach ensures that the learning process remains coherent and builds upon the strategies developed in previous sessions.

In Activity 2, students faced greater challenges compared to Activity 1 due to the increased complexity of adding more numbers, leading to longer group discussions. As anticipated in the HLT 1 conjecture, each group employed the skip counting strategy when calculating their weekly pocket money. When encountering difficulties, students requested toy money media, similar to their approach in Activity 1. This is evident from the following conversation with the "Cool" group:

Praha : Ustadzah, where is the money picture, we can't count this.
Teacher : Today there is no picture of money toy.
Azka : We must draw it ourselves, right?
Teacher : You can if you want.
Praha : There are so many pictures, is there any other way?

According to the teachers, the difficulties students experienced in Activity 2 are understandable, given that they are typically accustomed to simpler addition tasks involving only up to three numbers. Despite these challenges, the teachers noted a positive aspect: even though students faced difficulties, they remained persistent. With guidance from both researchers and teachers, the students successfully achieved the activity's goals.

Reyhana : Ustadzah, is this right? 20-8 is 12, so 32-8 is what, ust? (Almost leads to friendly numbers)
Teacher : Hmm, Reyhana is smart but unfortunately, she hasn't found the result yet.
Reyhana : Well, what do we do, ustadzah?
Teacher : Try to think calmly about how to make 15 + 12 as easy as 20 + 12.
Kennard : We can just change the number to 20 or 30 heheheheh.
Teacher : How do you change 15 to 20 Kennard?
Kennard : Like reyhana earlier 20 - 5.

A notable outcome of this activity was that two groups—the "Smart" group and the "Cool" group—demonstrated the ability to apply multiple strategies to solve a single problem. Specifically, these groups were able to use both the friendly number and splitting strategies. This indicates a significant development in their problem-solving skills and their ability to adapt and employ different mental strategies for more complex calculations.

Teacher : Almost right.
Keke : or 15 + 5.
Researcher : That's right.
Keke : So, 15 plus 5 ust huh?
Researcher : Yes, Keke.
Keke : If 12 + 12 is also 12 made into 20, right?
Kennard : Ustadzah, for a moment, did you see those 12 + 12 results in 24, right?
Teacher : Right, how do you know?
Kennard : I just multiplied it ust hehe it turns out the result is the same if the 12 is broken down.
Teacher : What does break down mean?
Kennard : You see, ust 1 + 1 is 2 and 2 + 2 is 4? Right?
Teacher : Exactly, but it shouldn't be 1 + 1 but 10 + 10, 12 is 10 and 2.
Kennard : Yes, that's the point.

The researchers were pleasantly surprised by the outcomes of Activity 2, as students exceeded their expectations. This success can be attributed to the open-ended questions posed by the researchers, which encouraged students to explore and apply various strategies creatively when calculating the amount of pocket money. The students were able to effectively explain their strategies during their presentations at the end of the meeting, demonstrating a clear understanding of their methods.

Moreover, students concluded that alms are not a one-time activity per week but should be practiced daily, reflecting a deeper grasp of the concept. Teachers noted that this approach to learning mathematics not only increased students' enthusiasm and persistence but also made the learning process more meaningful. The positive impact of incorporating a spiritual context, such as alms, alongside mathematical learning contributed to both the students' cognitive and spiritual development.

In Activity 3, the researchers' predictions from HLT 1 were largely confirmed. The "Great" and "Cool" groups encountered challenges when calculating both the total amount of pocket money they desired and the remaining amount after giving alms. Despite these difficulties, students effectively utilized the friendly number and splitting strategies they had learned. This observation aligns with the conjectures outlined in HLT 1, demonstrating that students were applying the mental strategies consistently across activities.

Teacher : Okay "Great" group, you want to discuss to count whose pocket money first?
Azza : It's easy, you can count who's pocket money first. We already know how because you taught us yesterday.
Marsya : Yes, ust, we can use the method from yesterday, right?
Teacher : You can try it first.

Researchers and teachers can conclude that children can capture and understand the results of the past day's learning, although the terms they use are different namely breakdown strategy and 20 make strategy.

Keke : Ust, I'm sorry, 15 + 15 can't be calculated by splitting ust, what do I do? Must be made into 20, right ust?
Teacher : How can we make it easy? You wrote it horizontally earlier, try writing it down.

sometime later
Kennard : The result is 30 right ust?
Teacher : How come it's so fast?
Reyhana : Yes, ust, if I'm not mistaken, ustadzah said that if the result is two numbers, the other number is kept up here.
Kennard : My teacher also taught me this ust in the past.
Teacher : Great.

In Activity 3, each group encountered challenges when adding numbers whose sums exceeded 10. Students reported difficulty applying the splitting calculation strategy for these problems due to the presence of tens in the results. Additionally, calculating the remaining money posed challenges, as anticipated. Some groups attempted to solve these problems by arranging the numbers vertically, but they struggled with the larger numbers they needed to subtract. This reflects the complexities involved in mental calculations with larger figures and highlights areas where further support may be needed.

Praha : Ust we've all counted the rest of my pocket money and Azka's pocket money, but we can't count Rani's pocket money.
Teacher : Why can't you count Rani's pocket money?
Teacher : How come?
Rani : Yes ust, Rani's pocket money is 50,000 and Rani's *infaq* is 15,000, so if we calculate 50-15, it is not possible.

Despite the difficulties, students demonstrated an understanding of how to calculate the remaining money using the borrowing strategy, as anticipated. They were able to grasp the concept and apply it in their calculations. Interestingly, the terminology students used aligned well with the established methods, indicating a clear comprehension of the strategy. This shows that even with initial challenges, students can adapt and utilize appropriate strategies to solve arithmetic problems.

Teacher : Praha is smart, indeed 0 is the smallest number, but how can 0 be reduced by 5?
Rani : Is it borrowing the number in front of it?
Teacher : That's right.
Prague : Oooh I know the 0 is borrowing 1 huh? If it borrows 1, the result will be 5, right 10-5 =.
The teacher and researcher clap their hands
Azka : You're smart.
Rani : But the result is 45, is that right?
Teacher : No, the result is not 45.
Rani : After all, 5-4 is 1 right?
Teacher : Rani forgot, didn't she? What did you do with the 5 to subtract the 0?
Prague : The 5 is borrowed 1 right ust, meaning that the remaining 4 is left ust? So, if you borrow it, you also subtract it ust?
Teacher : That's right Praha, something borrowed like an eraser is also subtracted.
Rani : Yes, the result must be 35, right?

After addressing a few questions, each group successfully grasped how to calculate the amount of pocket money using the counting and saving strategy. Remarkably, when the researcher provided brief explanations, one or two students from each group quickly identified the appropriate strategy to use. The teacher observed that while students might have initially forgotten or struggled with the concept, the activity effectively refreshed and enhanced their understanding. This process not only reminded students of previously learned strategies but also helped refine their ability to apply these strategies effectively. The teacher was pleased with the outcome, recognizing that the activity served as a valuable reinforcement of students' calculation skills and strategic thinking.

In the retrospective analysis of HLT 1, it was concluded that students had effectively mastered the calculation strategies and met the goals outlined for each activity and the overall mathematical objectives. As a result, researchers decided to expand the number of activities to four to further enrich students' calculation strategies. A new mental strategy, the jumping strategy, was introduced in HLT 2. This strategy involves students jumping to the nearest ten before splitting, aiming to facilitate the use of friendly numbers and enhance the splitting strategy.

In HLT 2, the jumping strategy involves students first rounding to the nearest ten, with the aim of facilitating the use of friendly numbers in subsequent calculations. This strategy also incorporates splitting, where students divide larger numbers into more manageable parts before applying the jumping strategy. By doing so, researchers hope to enhance students' ability to use these strategies effectively. Additionally, new activities have been introduced with varied alms contexts, both building on and diverging from those in HLT 1. These new activities are designed to provide more complex and concrete scenarios, encouraging students to gain a deeper understanding of the alms context and apply their mathematical strategies more broadly. The HLT for the second designed activity includes four distinct tasks aimed at deepening students' understanding and application of mathematical strategies within various contexts, namely “Let's get used to alms every day,” “Don't forget to give charity regularly this week,” “You can set your allowance by yourself,” and “Let's Help Pak Samad to break the fast.” Furthermore, Table 2 provides an overview of each activity's focus, the mathematical concepts addressed, and their contextual relevance.

Table 2. Summarizes the activities, HLT 1 Result, and mathematical goal the HLT 2

The Result of HLT 1	Activities	Mathematical Goal
We increased the activities into 4 activities to enrich the calculation strategy of students. The addition of the mental strategy is in the jumping calculation strategy in the second activity that was not previously in HLT 1. Jumping strategies are added so that students more easily lead to friendly numbers and splitting. This is in accordance with the sequence of counting strategies presented by Thompson, (1999, 2010)	Let's get used to alms every day	to perform addition and subtraction using skip counting calculation strategy
	Don't forget to give charity regularly this week	to perform addition and subtraction using jumping strategies
	You can set your allowance by yourself	to perform addition and subtraction using the friendly number and splitting strategies
	Let's Help Pak Samad to break the fast	perform addition and subtraction using the borrowing and saving strategies

Furthermore, researchers will discuss learning patterns in students when implementing the second HLT.

1. Let's get used to alms every day
   In this activity, the primary aim is to encourage students to use skip counting for summing amounts of money and to employ the skip counting back strategy to determine the remaining amount. This strategy is commonly observed among elementary students when counting objects and finding the remainder after subtraction (Thompson, 1999, 2000). The researcher deliberately selects specific monetary fragments for students to count, aiming to enhance their critical thinking about using fractions in calculations compared to their prior experience. Most groups utilize the counting-on strategy starting from larger amounts, with one group applying skip counting in patterns of 2.2 and 5.5. When prompted by the teacher to "How to count faster," students predominantly face challenges. Nonetheless, nearly three groups successfully implement the skip counting back strategy, while one group continues to rely on the counting back strategy from the largest number, counting one-by-one.
2. Don't forget to give charity regularly this week
   In comparing HLT 1 and HLT 2 for this activity, there are notable differences in their focus and goals. In HLT 1, the primary objective is to have students effectively use friendly numbers and splitting mental strategies for calculations. The approach emphasizes helping students become familiar with these strategies through guided activities. In contrast, HLT 2 introduces the jumping mental strategy as the central focus. This strategy involves using a number line to jump in tens and ones, making it easier for students to reach the correct answer. The goal in HLT 2 is for students to apply the jumping strategy to count their weekly allowance and determine the

remaining money after giving alms. This method is designed to enhance students' ability to handle addition and subtraction problems efficiently. Additionally, HLT 2 incorporates open-ended problems to encourage the emergence of new mental strategies. As students practice the jumping strategy, they are also prompted to use splitting techniques by dividing tens of thousands by thousands before performing jumps. This approach aims to deepen their understanding and application of the splitting strategy in subsequent activities. Overall, the jumping strategy in HLT 2 is expected to facilitate a better grasp of friendly numbers and enhance students' problem-solving skills in arithmetic. This progression from HLT 1 to HLT 2 is intended to support students in mastering addition and subtraction operations with greater ease (Beishuizen et al., 1997).

In this worksheet, students were tasked with recording their weekly allowance, which introduced opportunities for conflict and encouraged critical and creative thinking about their strategies. Two groups opted to use the skip counting on strategy, as taught previously, while the other two groups attempted to use multiplication. The latter approach, however, led to difficulties as students struggled to maintain their calculations midway. In response, the teacher provided guiding questions to help students explore and apply new strategies, such as jumping, friendly numbers, or splitting. Regarding the calculation of leftover money, most groups successfully applied the new strategies introduced in HLT 2—jumping, friendly numbers, or splitting. Only one group continued to use the skip counting back strategy. This indicates that students were able to engage with and understand new strategies through contexts that are relevant to their everyday lives, reflecting their ability to adapt and integrate these approaches into their problem-solving processes.

3. You can set your allowance by yourself

   In this activity, students are tasked with calculating their self-assigned weekly allowance, requiring group discussions to determine the most appropriate mental strategy. This activity is included in both HLT 1 and HLT 2, though the focus differs between the two. In HLT 1, the emphasis is on using carrying and borrowing mental strategies. Students generally grasp these strategies well, as they have previously learned to use splitting techniques. This observation aligns with Thompson's (1999) assertion that understanding regrouping or splitting facilitates the use of carrying and borrowing strategies. In contrast, HLT 2 directs students to employ friendly numbers and splitting mental strategies. While students initially struggle with the jumping strategy, they quickly adapt when guided by the teacher. This guidance helps them understand and effectively use friendly number counting and splitting strategies. When calculating the remaining allowance, students in HLT 1 often seek clarification on the carrying and borrowing methods, expressing doubts about their accuracy. In HLT 2, however, students can more readily and accurately apply friendly numbers and splitting strategies to determine the remaining money, demonstrating a clearer understanding and more efficient problem-solving approach.

4. Let's Help Pak Samad to break the fast

   This activity, exclusive to HLT 2, focuses on calculating the total amount of alms collected by all groups over a week and determining the remaining alms after contributing to Mr. Samad. It involves using the carrying strategy for addition and the borrowing strategy for subtraction, designed to present more complex and concrete problems to enhance student understanding of alms. At the start of the activity, many students resort to splitting and friendly number strategies. They struggle particularly with adding large numbers, such as 78,000 + 95,000. The teacher provides guiding questions to help students transition to using the carrying strategy effectively. For calculating the remaining alms, most students quickly and correctly apply the borrowing strategy without needing extensive teacher intervention. They generally seek confirmation to ensure their methods are accurate. This demonstrates that, despite initial difficulties, students are able to adapt to and utilize the borrowing strategy efficiently with minimal guidance.

In the design of the Hypothetical Learning Trajectory (HLT) outlined above, notable differences can be observed between the first and second iterations of the HLT. In the second iteration, an additional activity was incorporated to further enhance the mental strategies that students are expected to learn. These modifications were informed by the findings and retrospective analysis of HLT 1, the initial design of the HLT. This approach aligns with Mariana's (2009) assertion that conclusions and evidence derived from the analysis of previous HLT iterations can serve as valuable references for enriching subsequent HLT activities and hypotheses.

The retrospective analysis of HLT 1 revealed that students had successfully mastered the overall arithmetic strategies and achieved the goals set for both the activities and the mathematical objectives defined at the outset. Consequently, the researcher introduced a fourth activity in HLT 2 to further develop the students' mental strategies. Notably, the addition of a "jumping strategy" was a significant enhancement in HLT 2 that was not present in HLT 1. The jumping strategy involves guiding students to first "jump" to the nearest ten, facilitating the use of friendly numbers and aiding in the splitting process. This method aims to help students more effectively reach friendly numbers and apply splitting strategies.

Additionally, the new activities incorporate a more complex and concrete context of almsgiving (*infaq*), which was absent in HLT 1. While HLT 1 allowed students to understand the basic context of alms, the new activities with varied and intricate contexts are designed to deepen students' awareness and comprehension of the almsgiving context.

In summary, the researcher concludes that students are able to effectively learn addition and subtraction operations involving both monetary values and regular numbers through the use of these HLTs. The current HLT framework could be further enhanced by introducing additional strategies that students can master in arithmetic and by extending the duration of learning activities. A supportive classroom environment, characterized by direct discussions and the use of concrete materials, is crucial for optimizing the outcomes of the HLT. Moreover, the contexts and values incorporated into the activities do not necessarily need to focus on almsgiving but can be tailored to reflect other relevant aspects of the students' everyday lives (Beentjes & Jonker, 1987).

An overview of the HLT for teaching addition and subtraction involving money for second-grade elementary students is presented in Figure 2. This figure illustrates the sequential order of each strategy implemented in the activities. The strategies are categorized from basic skip counting and jumping strategies to more advanced techniques such as friendly numbers, splitting, and borrowing. Each strategy description in the left-hand box is accompanied by corresponding student work examples in the right-hand box, as detailed in Table 3.

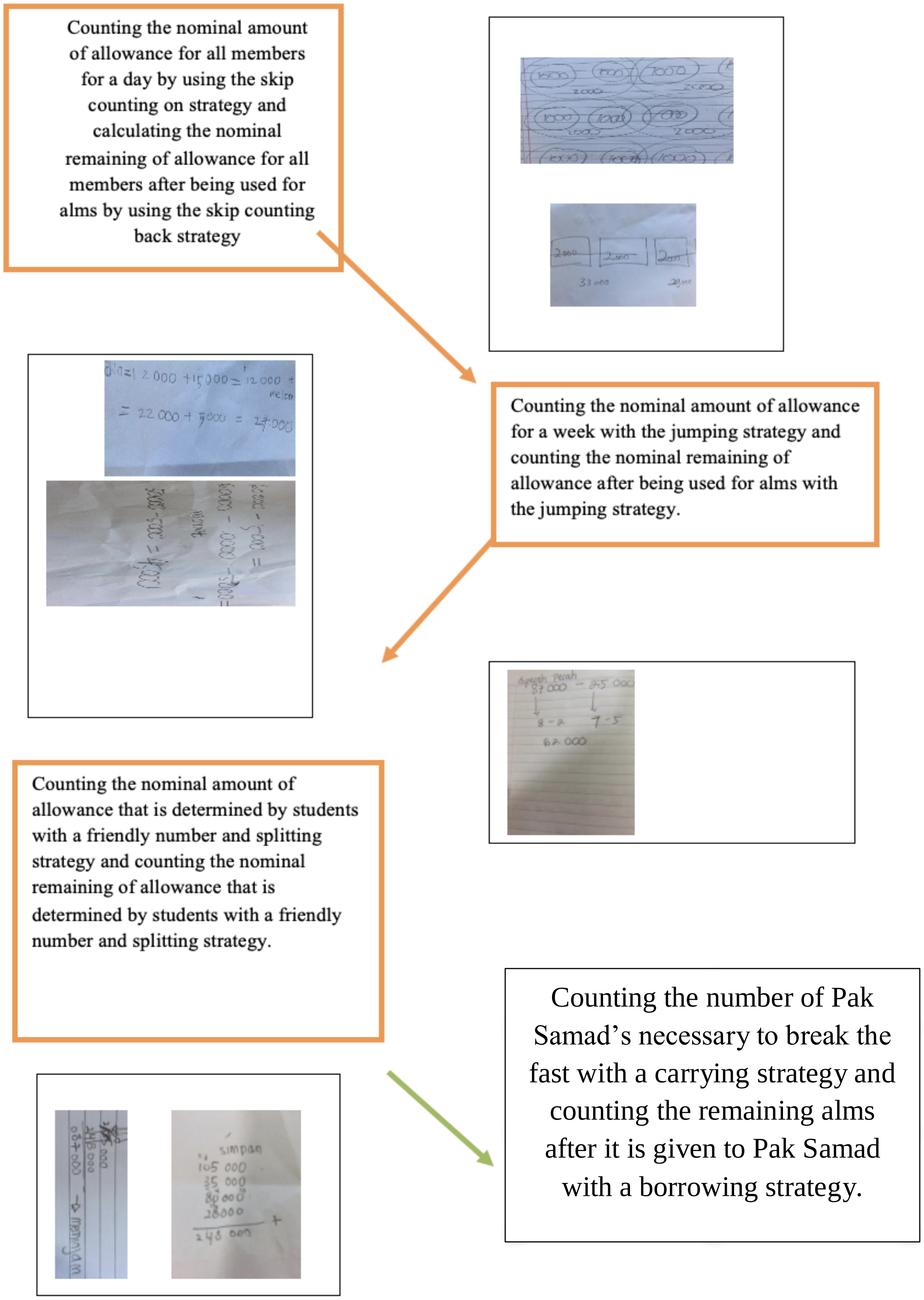

Figure 2. Schematic Overview of the Reconstructed HLT in Grade 2

Based on the retrospective analysis of HLT 1 and HLT 2, several strategies emerged from students during each activity, as illustrated in Figure 2. These strategies are described in more detail in Table 3.

Table 3. Emerging Mental Strategies in Each Designed Activity

Activity Name	Mathematical Activities	Emerging Mental Strategies
"Let's get used to alms everyday"	Calculate the nominal amount of allowance that all members have for a day	Counting on Counting on from larger Skip counting on
"Let's get used to alms everyday"	Calculate the remaining nominal allowance of all group members after the allowance is issued for alms	Counting back from larger Skip counting back
"Don't forget alms this week"	Calculate the nominal amount of each group member's allowance for a week	Skip counting on Friendly number Splitting strategy Jumping strategy
"Don't forget alms this week"	Calculate the remaining nominal money of each group member after being used for alms	Friendly number Splitting strategy Jumping strategy
"Set your own pocket money"	Calculates the nominal amount of each member's self-determined group allowance for a week	Friendly number strategy Splitting strategy Jumping strategy saving strategy
"Set your own pocket money"	Calculate the remaining nominal allowance of each member of the self-determined group for a week after it is used for alms	Friendly number strategy Splitting strategy Jumping strategy Borrowing strategy
"Let' s give away the proceeds of your money to break the fast Mr. Samad"	Calculate the nominal number of alms collected by one class	Friendly number strategy Splitting strategy Saving strategy
"Let' s give away the proceeds of your money to break the fast Mr. Samad"	Calculate the nominal amount needed to meet the needs of Mr. Samad	Saving strategy
"Let' s give away the proceeds of your money to break the fast Mr. Samad"	Calculating the remaining nominal alms class money after being used to spend Mr. Samad's needs for breaking fast	Friendly number strategy Splitting strategy Borrowing strategy

At the conclusion of the study involving almsgiving contexts, students demonstrated increased awareness and responsibility regarding their money management. Although the design could potentially be adapted to general monetary contexts, such as buying and selling, the specific focus on almsgiving appeared to enhance students' consciousness about their spending habits. On the final day of the study, one student remarked that it was feasible to perform acts of charity daily, given their regular pocket money. Instead of using their money for snacks or drinks provided by the school, the student chose to allocate their funds towards charitable activities. This shift in perspective reflects a growing sense of responsibility among the students, aligning with the school's Islamic values aimed at fostering piety, social awareness, and responsible behavior in students.

CONCLUSION

The designed activities for enhancing mathematical skills through the context of almsgiving values involve a series of four sequential tasks. The first activity, "Let's Get Used to Almsgiving Every Day," requires students to calculate the total allowance for a group using the skip counting strategy and then determine the remaining pocket money after performing almsgiving, employing the skip counting back strategy. The second activity, "Don't Forget to Give Charity Regularly This Week," involves students discussing and calculating their weekly allowance and the remainder after allocating funds for almsgiving throughout the week. In the third activity, "Determine Your Allowance," each group builds on the previous calculations by establishing their desired weekly allowance and the remaining amount after partially donating it. Finally, the fourth activity, "Let's Help Pak Samad to Break the Fast," involves students calculating the total alms collected to assist Mr. Samad with his fasting needs. Students read a story about Mr. Samad, calculate the funds required for his needs based on a provided list, and determine if the collected amount is sufficient. They are also tasked with deciding how to utilize any remaining funds. These activities collectively aim to integrate mathematical learning with practical applications of almsgiving values.

During the implementation of this almsgiving value-based learning trajectory, six calculation strategies emerged from the students: counting on, counting back, skip counting on, skip counting back, jumping counting, friendly numbers, splitting, and saving and borrowing strategies. The primary mathematical objectives of the four activities were to enable students to accurately determine the amount of money using carrying and borrowing strategies and to calculate remaining amounts using these strategies. Each activity was designed with distinct goals to help students develop various arithmetic strategies and achieve the overarching mathematical objectives.

However, the study has limitations, particularly in its focus on addition and subtraction. The scope was intentionally narrowed to align with the capabilities of second-grade students in elementary school, leaving out other operations such as multiplication and division. Future research could extend this context to include different mathematical topics and grade levels, thereby broadening the application and impact of the almsgiving value-based approach in mathematical education.

Acknowledgments

I am gratefully thankful to Center of Research at Universitas Negeri Surabaya.

Declarations

Author Contribution	:	NM: Conceptualization, Funding acquisition, Methodology, Project administration, Formal analysis, Investigation, Supervision, Validation, and Writing-review and editing.
		FES: Conceptualization, Data curation, Formal analysis, Investigation, Resources, Visualization, and Writing-original draft.

Funding Statement	:	This research and publication funding are supported by Universitas Negeri Surabaya
Conflict of Interest	:	The authors declare no conflict of interest.
Additional Information	:	There is no additional information in this paper

REFERENCES

Armanto, D. (2002). *Teaching multiplication and division realistically in Indonesian primary schools: A prototype of local instructional theory*. University of Twente. https://ris.utwente.nl/ws/portalfiles/portal/6085178/thesis_Armanto.pdf

Ashcraft, M. H. (1985). Is it farfetched that some of us remember our arithmetic facts?. *Journal for Research in Mathematics Education, 16*(2), 99-105. https://doi.org/10.5951/jresematheduc.16.2.0099

Ashcraft, M. H., & Fierman, B. A. (1982). Mental addition in third, fourth, and sixth graders. *Journal of Experimental Child Psychology, 33*(2), 216–234. https://doi.org/10.1016/0022-0965(82)90017-0

Baroody, A. J. (1992). The development of kindergartners' mental-addition strategies. *Learning and Individual Differences, 4*(3), 215–235. https://doi.org/10.1016/1041-6080(92)90003-W

Beentjes, J. W. J., & Jonker, V. H. (1987). Inconsistency in addition and subtraction strategies. *The Journal of Experimental Education, 56*(1), 4–7. https://doi.org/10.1080/00220973.1987.10806459

Beishuizen, M. (1993). Mental Strategies and Materials or Models for Addition and Subtraction up to 100 in Dutch Second Grades. *Journal for Research in Mathematics Education, 24*(4), 294-323. https://doi.org/10.2307/749464

Beishuizen, M., Van Putten, C. M., & Van Mulken, F. (1997). Mental arithmetic and strategy use with indirect number problems up to one hundred. *Learning and Instruction, 7*(1), 87–106. https://doi.org/10.1016/S0959-4752(96)00012-6

Callingham, R. (2005). Primary students' mental computation: Strategies and achievement. *Building Connections: Research, Theory and Practice*, 193–200. https://hdl.handle.net/1959.11/11373

Carpenter, T. P. (2013). Conceptual knowledge as a foundation for procedural knowledge. In *Conceptual and procedural knowledge* (pp. 113–132). Routledge. https://doi.org/10.4324/9780203063538

Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for Research in Mathematics Education, 29*(1), 3–20. https://doi.org/10.5951/jresematheduc.29.1.0003

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15*(3), 179–202. https://doi.org/10.5951/jresematheduc.15.3.0179

Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education, 37*(2), 87–115. https://doi.org/10.2307/30034843

Clements, D. H., & Sarama, J. (2012). Learning trajectories in mathematics education. In *Hypothetical Learning Trajectories* (pp. 81–90). Routledge. https://doi.org/10.4324/9780203063279

Cooke, H. (2001). *Primary Mathematics* (Vol. 2). SAGE.

Crooks, N. M., & Alibali, M. W. (2014). Defining and measuring conceptual knowledge in mathematics. *Developmental Review, 34*(4), 344–377. https://doi.org/10.1016/j.dr.2014.10.001

Freudenthal, H. (2012). *Mathematics as an educational task*. Springer Science & Business Media. https://doi.org/10.1007/978-94-010-2903-2

Geary, D. C., Hoard, M. K., Byrd-Craven, J., & DeSoto, M. C. (2004). Strategy choices in simple and complex addition: Contributions of working memory and counting knowledge for children with mathematical disability. *Journal of Experimental Child Psychology, 88*(2), 121–151. https://doi.org/10.1016/j.jecp.2004.03.002

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning, 6*(2), 105–128. https://doi.org/10.1207/s15327833mtl0602_3

Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In *Educational design research* (pp. 29–63). Routledge. https://doi.org/10.4324/9780203088364

Gravemeijer, K., & Terwel, J. (2000). Hans Freudenthal: A mathematician on didactics and curriculum theory. *Journal of Curriculum Studies, 32*(6), 777–796. https://doi.org/10.1080/00220270050167170

Gravemeijer, K., & van Eerde, D. (2009). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *The Elementary School Journal, 109*(5), 510–524. https://doi.org/10.1086/596999

Heirdsfield, A. (2011). Teaching mental computation strategies in early mathematics. *YC Young Children, 66*(2), 96–102. https://eprints.qut.edu.au/42451/1/42451.pdf

Herman, T. (2001). Strategi mental yang digunakan siswa sekolah dasar dalam berhitung. *Seminar Nasional Pendidikan Matematika di Universitas Negeri Yogyakarta*, 1–7.

Hiebert, J., & Lefevre, P. (2013). Conceptual and procedural knowledge in mathematics: An introductory analysis. In *Conceptual and procedural knowledge* (pp. 1–27). Routledge. https://doi.org/10.4324/9780203063538

Hopf, C. (2004). Qualitative interviews: An overview. *A Companion to Qualitative Research, 203*(8), 100093.

Karlsson, C., & Johansson, E. (2018). *MATEMATIKUNDERVISNING I SUBTRAKTION: En litteraturstudie om undervisning, metoder och strategier i subtraktion*. https://www.diva-portal.org/smash/get/diva2:1240818/FULLTEXT02

Macintyre, T., & Forrester, R. (2003). Strategies for mental calculation. *Proceedings of the British Society for Research into Learning Mathematics, 23*(2), 49–54. https://www.nuffieldfoundation.org/wp-content/uploads/2019/11/Article_STRATEGIES20FOR20MENTAL20CALCULATION_2003.pdf

Mariana, N. (2009). *Design research in geometry education developing spatial abilities in first grade children.* Utrecht University. https://www.fisme.science.uu.nl/en/impome/theses_group_2007/thesis_Neni.pdf

Mariana, N. (2019). Transformation of research education at elementary school mathematics in the industry 4.0. *International Journal of Innovation, Creativity and Change, 5*(5), 1–10. https://www.ijicc.net/images/vol5iss5/Part_2/55223_Mariana_2020_E_R.pdf

Mariana, N., Firmandani, F. A., & Indrawati, D. (2018). Labour Issues as a Context in Designing Mathematics Activity for Counting Strategy. *MISEIC 2018*.

McIntosh, A., Reys, B. J., & Reys, R. E. (2005). A proposed framework for examining basic number sense. In *Subject Learning in the Primary Curriculum* (pp. 209–221). Routledge. https://doi.org/10.4324/9780203990247

Moelong, L. J. (2012). *Metode Penelitian Kualitatif*. Remaja Rosda Karya.

Nengsih, G. A., & Pujiastuti, H. (2021). Analisis kesulitan dalam menyelesaikan soal materi operasi bilangan cacah siswa sekolah dasar. *JKPM (Jurnal Kajian Pendidikan Matematika), 6*(2), 293-306. http://dx.doi.org/10.30998/jkpm.v6i2.9941

Prahmana, R. C. I. (2017). *Design Research (Teori dan Implementasinya: Suatu Pengantar)*. Rajawali Press.

Prahmana, R. C. I., & Kusumah, Y. S. (2017). Didactic trajectory of research in mathematics education using research-based learning. *Journal of Physics: Conference Series, 893*(1), 012001. http://dx.doi.org/10.1088/1742-6596/893/1/012001

Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes: An overview on achievements and challenges. *ZDM, 47*, 877–891. https://doi.org/10.1007/s11858-015-0722-3

Principles, N. (2000). *Standards for School Mathematics*. NCTM.

Proulx, J. (2013). Mental mathematics, emergence of strategies, and the enactivist theory of cognition. *Educational Studies in Mathematics, 84*, 309–328. https://doi.org/10.1007/s10649-013-9480-8

Proulx, J. (2019). Mental mathematics under the lens: Strategies, oral mathematics, enactments of meanings. *The Journal of Mathematical Behavior, 56*, 100725. https://doi.org/10.1016/j.jmathb.2019.100725

Resnick, L. B. (1986). The development of mathematical intuition. *Minnesota Symposia on Child Psychology, 19*, 159–194.

Reys, B. J. (1985). Mental computation. *The Arithmetic Teacher, 32*(6), 43–46. https://doi.org/10.5951/AT.32.6.0043

Reys, B. J. (1986). Estimation and Mental Computation It's "About" Time. *The Arithmetic Teacher, 34*(1), 22–23. https://doi.org/10.5951/AT.34.1.0022

Reys, B. J., Reys, R. E., & Hope, J. A. (1993). Mental computation: A snapshot of second, fifth and seventh grade student performance. *School Science and Mathematics, 93*(6), 306–315. https://doi.org/10.1111/j.1949-8594.1993.tb12251.x

Reys, R. E., Reys, B. J., Nohda, N., & Emori, H. (1995). Mental computation performance and strategy use of Japanese students in grades 2, 4, 6, and 8. *Journal for Research in Mathematics Education, 26*(4), 304–326. https://doi.org/10.5951/jresematheduc.26.4.0304

Risdiyanti, I., & Prahmana, R. C. I. (2020). The Learning Trajectory of Number Pattern Learning Using "Barathayudha" War Stories and Uno Stacko. *Journal on Mathematics Education, 11*(1), 157–166. https://doi.org/10.22342/jme.11.1.10225.157-166

Simon, M. A., & Tzur, R. (2012). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. In *Hypothetical learning trajectories* (pp. 91–104). Routledge. https://doi.org/10.4324/9780203063279

Siswono, T. Y. E. (2006). PMRI: Pembelajaran Matematika yang Mengembangkan Penalaran, Kreativitas dan Kepribadian Siswa. *Makalah Workshop Pembelajaran Matematika di MI "Nuruh Rohmah"*. Sidoarjo, 8.

Soedjadi, R. (2014). Inti Dasar – Dasar Pendidikan Matematika Realistik Indonesia. *Jurnal Pendidikan Matematika, 1*(2), 1–10. https://doi.org/10.22342/jpm.1.2.807.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher, 41*(5), 147–156. https://doi.org/10.3102/0013189X12442801

Thevenot, C., Barrouillet, P., Castel, C., & Uittenhove, K. (2016). Ten-year-old children strategies in mental addition: A counting model account. *Cognition, 146*, 48–57. https://doi.org/10.1016/j.cognition.2015.09.003

Thompson, I. (1999). Mental calculation strategies for addition and subtraction: Part 1. *Mathematics in School, 28*(5), 2–4. https://doi.org/10.1017/CBO9781107415324.004

Thompson, I. (2000). Mental calculation strategies for addition and subtraction: Part 2. *Mathematics in School, 29*(1), 24–26. https://www.jstor.org/stable/30212066

Thompson, I. (2010). 12 Getting your head around mental calculation. *Issues in Teaching Numeracy in Primary Schools*, 161-173.

Treffers, A. (1991). Didactical background of a mathematics program for primary education. *Realistic Mathematics Education in Primary School*, 21–56.

Van den Akker, J., Gravemeijer, K., McKenney, S., & Nieveen, N. (2006). *Educational Design Research*. Routledge. https://doi.org/10.4324/9780203088364

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. *Encyclopedia of Mathematics Education*, 713–717. https://doi.org/10.1007/978-3-030-15789-0_170

Van Nes, F., & De Lange, J. (2007). Mathematics education and neurosciences: Relating spatial structures to the development of spatial sense and number sense. *The Mathematics Enthusiast, 4*(2), 210–229. https://doi.org/10.54870/1551-3440.1072

Varol, F., & Farran, D. (2007). Elementary school students' mental computation proficiencies. *Early Childhood Education Journal, 35*, 89–94. https://doi.org/10.1007/s10643-007-0173-8

Widaman, K. F., Geary, D. C., Cormier, P., & Little, T. D. (1989). A componential model for mental addition. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 15*(5), 898-919. https://doi.org/10.1037/0278-7393.15.5.898

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458-477. https://doi.org/10.5951/jresematheduc.27.4.0458